

A GEOMETRIC SETTING FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

IOAN BUCATARU, OANA CONSTANTINESCU, AND MATIAS F. DAHL

ABSTRACT. To a system of second order ordinary differential equations (SODE) one can assign a canonical nonlinear connection that describes the geometry of the system. In this work we develop a geometric setting that allows us to assign a canonical nonlinear connection also to a system of higher order ordinary differential equations (HODE). For this nonlinear connection we develop its geometry, and explicitly compute all curvature components of the corresponding Jacobi endomorphism. Using these curvature components we derive a Jacobi equation that describes the behavior of nearby geodesics to a HODE. We motivate the applicability of this nonlinear connection using examples from the equivalence problem, the inverse problem of the calculus of variations, and biharmonicity. For example, using components of the Jacobi endomorphism we express two Wuenschmann-type invariants that appear in the study of scalar third or fourth order ordinary differential equations.

1. INTRODUCTION

It is well known that to a system of second order ordinary differential equations (SODE) one can associate a nonlinear connection in a canonical way. This nonlinear connection further induce various geometric objects useful for the study of the SODE such as the dynamical covariant derivative, parallel transport, torsions, curvatures, and the Jacobi equation [7, 17, 24, 25, 27, 35]. There are various attempts to define a similar canonical nonlinear connection also to systems of higher order ordinary differential equations (HODE), [5, 6, 18, 20, 31, 33, 34]. A difficulty with this task is that to a HODE one can associate a number of nonlinear connections, and each of these offer different information about the HODE, [1, 8, 9, 12, 16, 36].

The purpose of this work is to develop a unifying geometric setting for studying systems of second and higher order ordinary differential equations. Our approach is focused on two ideas of Kosambi [27, 28]. First, in [27], Kosambi states that a "system of differential equations can be dealt with geometrically by means of the tensorial operator of differentiation, called the bi-derivative". In this work we propose a global expression, called the dynamical covariant derivative, for the bi-derivative operator introduced by Kosambi. The dynamical covariant derivative we propose, is associated to a system of differential equations, of second or higher order, and a nonlinear connection that is not fixed yet. We determine the nonlinear connection by requiring the compatibility of the dynamical covariant derivative with some geometric structure, introduced by the equations of variation of the system of differential equations. This follows the statement of Kosambi, [28], that for a system of HODE, "the connection cannot be determined without the recourse to the equations of variation".

In Section 2 we review the geometrical setting for systems of second order ordinary differential equations. We describe the canonical nonlinear connection [17, 24], the dynamical covariant derivative [10, 14], symmetries, newtonoid vector fields [32, 40], the Jacobi endomorphism and the

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Jacobi equation [19, 25]. The purpose of this section is to present these objects such that their generalization to systems of higher order ordinary differential equations will be natural.

In Section 3 we start with a system of HODE, identified with a vector field S , which is called a semispray, on the k -th order tangent bundle $T^k M$ of the configuration manifold M . In this part of the work we show what are the advantages of using a specific nonlinear connection if we want to obtain information about the first order variation and the symmetries of a system of higher order differential equations. This connection has been proposed in a coordinate form by Miron and Atanasiu in [36] and has its origins in the work of Kosambi [28]. In Proposition 3.8 and Theorem 3.10 we fix this nonlinear connection, among the other ones that can be associated to a semispray of order k , using the following structures: the dynamical covariant derivative ∇ , the Jacobi endomorphism Φ , and an algebraic structure of the set of newtonoid vector fields of order k . In Theorem 3.11 we use the chosen nonlinear connection to characterize the set of symmetries of a semispray of order k using higher order dynamical covariant derivatives and the Jacobi endomorphism. We provide explicit formulae for all curvature components of the Jacobi endomorphism. In the last part of this section we discuss the first order geodesic variation and obtain the corresponding Jacobi equations for a semispray of order k .

In Section 4 we discuss the applicability of the theory developed in this paper in various fields, using examples from the equivalence problem, the inverse problem of the calculus of variation, and biharmonicity. In Subsection 4.1 we consider a second order Lagrangian L_2 that represents the second order prolongation of a Riemannian structure [9]. Its variational semispray is a semispray of order 3 and the corresponding Jacobi endomorphism encodes information about the curvature of the Riemannian structure, and their first and second order covariant derivatives. The second order Lagrangian L_2 has been used also in [13] to prove that biharmonic curves are variational. We further show that some invariants that have been associated to third or fourth order ordinary differential equations can be expressed in a geometrical way in terms of the curvature components of the Jacobi endomorphism and their dynamical covariant derivatives. In Subsection 4.2 we provide a geometric reformulation for one of these invariants, called the Wuenschmann invariant. The Wuenschmann invariant represents the obstruction for the existence of an associated conformal Lorentzian structure on the 3-dimensional solution space of a third order differential equation, [20, 23]. Also, it has been shown that the Wuenschmann invariant represents an obstruction for an equivalence problem of third order ordinary differential equations, [38]. In Subsection 4.3 we provide a geometric reformulation for an invariant that was proposed by Fels in [22] for characterizing variational fourth order ordinary differential equations. This invariant has been also considered by Dridi and Neut in [21] for studying the equivalence problem of fourth order ordinary differential equations.

2. SYMMETRIES AND FIRST ORDER VARIATION FOR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

In this section we reformulate some results about the geometry of a system of second order ordinary differential equations such that each of them has a generalization to system of higher order ordinary differential equations.

2.1. Systems of SODE and geometric structures on TM . Consider M a real, n -dimensional manifold, which is a topological Hausdorff space with countable base that is locally homeomorphic to \mathbb{R}^n , and with C^∞ -smooth transition maps. We assume that all objects are smooth where defined. For a manifold M , its tangent bundle (TM, π, M) is denoted by TM , the ring of smooth functions on M is denoted by $C^\infty(M)$, and the set of vector fields on M is denoted by $\mathfrak{X}(M)$. The canonical submersion of the tangent bundle, $\pi : TM \rightarrow M$, induces a natural foliation on TM , whose leaves are tangent spaces $\pi^{-1}(p) = T_p M$, for $p \in M$. Local coordinates on the base manifold

M are denoted by (x^i) , while induced coordinates on TM will be denoted by (x^i, y^i) . Consider also $TM \setminus \{0\} = \{(x, y) \in TM, y \neq 0\}$, the tangent bundle with the zero section removed.

Throughout the paper we will consider I a nonempty open interval in \mathbb{R} . For a curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, denote by $c' : I \rightarrow TM$, $c'(t) = (x^i(t), dx^i/dt)$, its tangent lift. We say that curve c is a *regular curve* if $c'(t) \in TM \setminus \{0\}$ for all $t \in I$.

The *tangent structure* (or vertical endomorphism) is the $(1, 1)$ -type tensor field on TM defined as $J = \partial/\partial y^i \otimes dx^i$. The *vertical distribution* is the n -dimensional distribution given by $V : u \in TM \mapsto V_u TM = \text{Ker } J_u = \text{Im } J_u = \text{Ker } D_u \pi \subset T_u TM$, where $D_u \pi$ denotes the tangent map of π at $u \in TM$. This distribution is integrable being tangent to the natural foliation of the tangent bundle.

In this section we pay attention to the geometry of a system of second order ordinary differential equations. Such a system of SODE on M can be represented using a semispray, which is a special vector field on $TM \setminus \{0\}$.

A *semispray* on M is a vector field $S \in \mathfrak{X}(TM \setminus \{0\})$ such that any integral curve of S , $\gamma : I \rightarrow TM \setminus \{0\}$, is of the form $\gamma = (\pi \circ \gamma)'$. The reason that a semispray S is only defined on $TM \setminus \{0\}$ and not on the whole tangent bundle TM is to include also semisprays induced by Finsler metrics [4, 43]. In induced coordinates (x, y) for $TM \setminus \{0\}$, a semispray S is given by

$$(2.1) \quad S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},$$

for some functions G^i defined on domains of (x, y) . For an integral curve $\gamma : I \rightarrow TM \setminus \{0\}$ of S , we say that curve $c : I \rightarrow M$, $c = \pi \circ \gamma$ is a *geodesic* of S . Therefore, a regular curve $c : I \rightarrow M$ is a geodesic of S if and only if $S \circ c' = c''$. Locally, a regular curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, is a geodesic of S if and only if it satisfies the system of second order ordinary differential equations

$$(2.2) \quad \frac{d^2 x^i}{dt^2} + 2G^i \left(x, \frac{dx}{dt} \right) = 0.$$

Thus a semispray describes systems of SODE with regular curves on M as solutions.

A geometric structure useful for the geometry of a system of SODE is that of nonlinear connection. An n -dimensional distribution $H : u \in TM \setminus \{0\} \mapsto H_u TM \subset T_u TM$ that is supplementary to the vertical distribution is called a *nonlinear connection*, or *horizontal distribution*. Therefore, for a nonlinear connection, we have the following decomposition: $T_u TM = H_u \oplus V_u$, for all $u \in TM \setminus \{0\}$. We will denote by h and v the horizontal and vertical projectors that correspond to the above decomposition, and since $h + v = \text{Id}$, either of these two projectors characterizes the nonlinear connection H . We will denote by $\mathfrak{X}^h(TM \setminus \{0\}) = h(\mathfrak{X}(TM \setminus \{0\}))$ and $\mathfrak{X}^v(TM \setminus \{0\}) = v(\mathfrak{X}(TM \setminus \{0\}))$ the $C^\infty(TM \setminus \{0\})$ -modules of horizontal and respectively vertical vector fields.

In induced coordinates (x^i, y^i) for TM , the vertical distribution is spanned by $\partial/\partial y^i$. Therefore, the horizontal distribution is locally spanned by vector fields of the form

$$\frac{\delta}{\delta x^i} = h \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - N_i^j(x, y) \frac{\partial}{\partial y^j},$$

for some functions N_j^i defined on domains of induced coordinates on $TM \setminus \{0\}$. The corresponding horizontal and vertical projectors can be written as

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes dy^i.$$

The 1-forms $\delta y^i = dy^i + N_j^i dx^j$ are annihilators for the horizontal distribution. A change of induced local coordinates $(x^i, y^i) \rightarrow (\tilde{x}^i(x), \tilde{y}^i(x, y))$ on TM induces the following transformation rule for

the basis $\delta/\delta x^i, \partial/\partial y^i$ and its dual basis $dx^i, \delta y^i$

$$\frac{\delta}{\delta x^j} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\delta}{\delta \tilde{x}^i}, \quad \frac{\partial}{\partial y^j} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial}{\partial \tilde{y}^i}, \quad d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \quad \delta \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \delta y^j.$$

Therefore the components of a tensor field on TM , with respect to these bases, will transform as the components of a tensor field on the base manifold M .

Although a semispray S always induces a canonical nonlinear connection, we will not yet assume any relation between S and (h, v) at this point. For a semispray S and an arbitrary nonlinear connection (h, v) , consider the vertically valued $(1, 1)$ -type tensor field on $TM \setminus \{0\}$,

$$\Phi = -v \circ \mathcal{L}_S v = v \circ \mathcal{L}_S h = v \circ \mathcal{L}_S \circ h,$$

which will be called the *Jacobi endomorphism*. Here, for a $(1, 1)$ -type tensor field A , we denote

$$\mathcal{L}_S A = \mathcal{L}_S \circ A - A \circ \mathcal{L}_S,$$

the Frölicher-Nijenhuis bracket of S and A . Locally, the Jacobi endomorphism can be expressed as follows

$$(2.3) \quad \Phi = R_j^i(x, y) \frac{\partial}{\partial y^i} \otimes dx^j, \quad R_j^i = 2 \frac{\delta G^i}{\delta x^j} - S(N_j^i) + N_k^i N_j^k.$$

The components R_j^i of the Jacobi endomorphism have been considered by Kosambi [27] as the second geometric invariant of the semispray. From this invariant one can determine two other geometric invariants, namely the curvature of the nonlinear connection and one component of the curvature of the Berwald connection [2]. In the Riemannian context, the components R_j^i of the Jacobi endomorphism are related to the curvature R_{jkl}^i , [37], of the Levi-Civita connection as follows $R_j^i(x, y) = R_{kjl}^i(x) y^k y^l$. A similar formula holds true also in the Finslerian context, [42, (8.15)], where R_j^i is called the Riemann curvature. For the Jacobi endomorphism Φ in formula (2.3), we will refer to its components as to the *curvature components*.

For a pair $(S, (h, v))$, consider the map $\nabla : \mathfrak{X}(TM \setminus \{0\}) \rightarrow \mathfrak{X}(TM \setminus \{0\})$, given by

$$(2.4) \quad \nabla = h \circ \mathcal{L}_S \circ h + v \circ \mathcal{L}_S \circ v = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v$$

that will be called the *dynamical covariant derivative*. By setting $\nabla f = S(f)$, for $f \in C^\infty(TM \setminus \{0\})$, using the Leibniz rule, and the requirement that ∇ commutes with tensor contraction, we extend the action of ∇ to arbitrary tensor fields and forms on $TM \setminus \{0\}$, see [10, Section 3.2]. For example, if ω is a 1-form on $TM \setminus \{0\}$, then its dynamical covariant derivative is given by

$$(2.5) \quad (\nabla \omega)(X) = S(\omega(X)) - \omega(\nabla X).$$

For a $(1, 1)$ -type tensor field A on $TM \setminus \{0\}$, its dynamical covariant derivative is given by

$$(2.6) \quad \nabla A = \nabla \circ A - A \circ \nabla.$$

From first formulae in (2.4) and (2.6) it follows that $\nabla h = 0$ and $\nabla v = 0$. Hence ∇ preserves the horizontal and vertical distributions H and V . However, ∇ does not act in a similar way on these two distributions. This can be seen locally as follows. Using formulae (2.4) and (2.5) we have

$$(2.7) \quad \begin{aligned} \nabla \frac{\delta}{\delta x^i} &= N_i^j \frac{\delta}{\delta x^j}, & \nabla dx^i &= -N_j^i dx^j, \\ \nabla \frac{\partial}{\partial y^i} &= \left(2 \frac{\partial G^j}{\partial y^i} - N_i^j \right) \frac{\partial}{\partial y^j}, & \nabla \delta y^i &= - \left(2 \frac{\partial G^i}{\partial y^j} - N_j^i \right) \delta y^j. \end{aligned}$$

The action of the dynamical covariant derivative ∇ on the components of a horizontal vector field $X = hX$ is given by

$$(2.8) \quad \nabla X = \nabla \left(X^i \frac{\delta}{\delta x^i} \right) = \nabla X^i \frac{\delta}{\delta x^i}, \quad \nabla X^i = S(X^i) + N_j^i X^j.$$

In formula (2.8) both components X^i and ∇X^i transform, under a change of induced coordinates on TM , as the components of a vector field from the base manifold M . The first order differential operator ∇X^i was introduced by Kosambi [27] with the name of bi-derivative. The term "dynamical covariant derivative" was introduced by Cariñena and Martinez in [14].

Next lemma gives some compatibility conditions between some of the geometric structures introduced so far. Let us emphasize that in the next lemma we do not yet assume any relation between the semispray S and the nonlinear connection (h, v) . However, we will use this lemma to fix a nonlinear connection in Proposition 2.7.

Lemma 2.1. *Consider a semispray S , a nonlinear connection (h, v) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h, v))$. Then,*

$$(2.9) \quad h \circ \mathcal{L}_S \circ J = -h, \quad J \circ \mathcal{L}_S \circ v = -v,$$

$$(2.10) \quad \mathcal{L}_S J + \text{Id} - 2v = \nabla J, \quad \nabla J = 2 \left(\frac{\partial G^j}{\partial y^i} - N_i^j \right) \frac{\partial}{\partial y^j} \otimes dx^i.$$

Proof. For an arbitrary vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ we have $X + [S, JX] \in \mathfrak{X}^v(TM \setminus \{0\})$. Composing from the left with h it follows that $hX = h[JX, S]$, which shows that first formula (2.9) is true. Similarly, composing by J , setting $JX = vZ$ for some $Z \in \mathfrak{X}(TM \setminus \{0\})$ gives the second equality in formula (2.9).

Using formula (2.4) and formulae (2.9) we obtain that

$$\begin{aligned} \nabla \circ J &= v \circ \mathcal{L}_S \circ v \circ J = v \circ \mathcal{L}_S \circ J = (\text{Id} - h) \circ \mathcal{L}_S \circ J = \mathcal{L}_S \circ J + h, \\ J \circ \nabla &= J \circ h \circ \mathcal{L}_S \circ h = J \circ \mathcal{L}_S \circ h = J \circ \mathcal{L}_S \circ (\text{Id} - v) = J \circ \mathcal{L}_S + v. \end{aligned}$$

Using the above two formulae and formula (2.6) we obtain that $\nabla J = \mathcal{L}_S J + \text{Id} - 2v$, which is formula (2.10). Second formula (2.10) follows directly from the local expressions (2.7) of the dynamical covariant derivative. \square

We note that from formula (2.10), the $(1, 1)$ -type tensor $\nabla J = \mathcal{L}_S J + \text{Id} - 2v$ represents the difference between an arbitrary nonlinear connection (h, v) , which is not fixed yet, and the canonical nonlinear connection associated to a semispray S that will be fixed in Proposition 2.7.

2.2. Symmetries for systems of SODE. It is well known that a semispray induces a canonical nonlinear connection which in turn, determines the five geometric invariants of the semispray. This is known as KCC theory, after Kosambi, Cartan, and Chern [2]. In this section we show that this connection can be fixed using symmetries of the semispray. This approach follows the statement of Kosambi [28] "the connection cannot be determined without recourse to the equations of variation". In the next section we show that this approach of fixing a connection also generalize to systems of higher order differential equations.

Definition 2.2. A vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ is a *dynamical symmetry* of a semispray S if $[S, X] = 0$.

A direct calculation shows that a vector field on $TM \setminus \{0\}$, locally expressed as

$$(2.11) \quad X = X^i(x, y) \frac{\partial}{\partial x^i} + Y^i(x, y) \frac{\partial}{\partial y^i},$$

is a dynamical symmetry if and only if $Y^i = S(X^i)$ and

$$(2.12) \quad S^2(X^i) + X(2G^i) = 0.$$

It follows that, while studying dynamical symmetries for a semispray S , the following set of vector fields on $TM \setminus \{0\}$ plays an important role:

$$(2.13) \quad \mathfrak{X}_S^1 = \left\{ X \in \mathfrak{X}(TM \setminus \{0\}), X = X^i \frac{\partial}{\partial x^i} + S(X^i) \frac{\partial}{\partial y^i} \right\}.$$

A vector field $X \in \mathfrak{X}_S^1$ is called a *newtonoid* [32]. Without local coordinates, the set of newtonoid vector fields can be expressed as follows

$$(2.14) \quad \mathfrak{X}_S^1 = \text{Ker}(J \circ \mathcal{L}_S) = \text{Im}(\text{Id} + J \circ \mathcal{L}_S).$$

For a vector field $X = X^i \partial / \partial x^i \in \mathfrak{X}(M)$ the *complete lift*, $X^{1,1} \in \mathfrak{X}(TM \setminus \{0\})$, is the vector field defined by

$$(2.15) \quad X^{1,1} = X^i(x) \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}(x) y^j \frac{\partial}{\partial y^i}.$$

The *vertical lift* of a vector field $X \in \mathfrak{X}(M)$ is the vector field $X^{1,0} \in \mathfrak{X}(TM \setminus \{0\})$ given by $X^{1,0} = J(X^{1,1})$ [45]. We denote by $\mathfrak{X}^{1,1}(TM \setminus \{0\})$, the set of complete lifts of vector fields on M , and by $\mathfrak{X}^{1,0}(TM \setminus \{0\}) = J(\mathfrak{X}^{1,1}(TM \setminus \{0\}))$, the set of vertical lifts of vector fields on M .

From expression (2.13) and formula (2.15) it follows that complete lifts and newtonoid vector fields are related by

$$\mathfrak{X}^{1,1}(TM \setminus \{0\}) \subseteq \bigcap_{S \text{ semispray}} \mathfrak{X}_S^1,$$

and equality holds if and only if $\dim M \geq 2$. If $\dim M = 1$ the above inclusion is strict due to the fact that the fibers of $TM \setminus \{0\}$ are not connected.

Definition 2.3. A vector field $X \in \mathfrak{X}(M)$ is a *Lie symmetry* of a semispray S if its complete lift, $X^{1,1}$, is a dynamical symmetry, which means that $[S, X^{1,1}] = 0$.

Next, we define a C^∞ -module structure on the set of newtonoids. This module structure was introduced in [40] for discussing adjoint symmetries for systems of ordinary differential equations. In proposition 2.7 we show that this structure can be used to characterize the canonical nonlinear connection associated to a semispray.

Remark 2.4. For $f \in C^\infty(TM \setminus \{0\})$ and $X \in \mathfrak{X}_S^1$, we define the product

$$f * X = (\text{Id} + J \circ \mathcal{L}_S)(fX) = fX + S(f)JX.$$

- i) With respect to the product $*$, the set \mathfrak{X}_S^1 has the structure of a $C^\infty(TM \setminus \{0\})$ -module, and the set $\mathfrak{X}^{1,1}(TM \setminus \{0\})$ has the structure of a $C^\infty(M)$ -module.
- ii) The maps $J : (\mathfrak{X}_S^1, *) \rightarrow (\mathfrak{X}^v(TM \setminus \{0\}), \cdot)$ and $h : (\mathfrak{X}_S^1, *) \rightarrow (\mathfrak{X}^h(TM \setminus \{0\}), \cdot)$ are isomorphisms between $C^\infty(TM \setminus \{0\})$ -modules. The map $D\pi : (\mathfrak{X}^{1,1}(TM \setminus \{0\}), *) \rightarrow (\mathfrak{X}(M), \cdot)$ is an isomorphism between $C^\infty(M)$ -modules.
- iii) A vector field X on $TM \setminus \{0\}$ is a newtonoid if and only if it can be expressed as follows

$$X = X^i(x, y) * \frac{\partial}{\partial x^i}.$$

We have seen that a vector field X on $TM \setminus \{0\}$ is a dynamical symmetry if and only if it is a newtonoid and satisfies equation (2.12). Therefore, a vector field $X = X^i(x) \partial / \partial x^i$ on M is a Lie symmetry if and only if components X^i satisfy the system of equations (2.12). Our aim now is to rewrite equations (2.12) such that its terms will have a covariant character. Note that neither

$S^2(X^i)$, nor $X(2G^i)$ in formula (2.12), has such a covariant character. For this we evaluate first the horizontal and vertical components of a vector field, which behave as the components of a vector field from the base manifold. This will allow us to characterize newtonoid vector fields using the dynamical covariant derivative.

Lemma 2.5. *Consider a semispray S , a nonlinear connection (h, v) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h, v))$. A vector field X on $TM \setminus \{0\}$ is a newtonoid if and only if $v(X) = J(\nabla X)$, which locally is equivalent to*

$$(2.16) \quad X = X^i \frac{\delta}{\delta x^i} + \nabla X^i \frac{\partial}{\partial y^i},$$

for some functions X^i defined on the domain of induced coordinates on $TM \setminus \{0\}$, where ∇X^i is defined in formula (2.8).

Proof. Since $J \circ \nabla = J \circ \mathcal{L}_S + v$ it follows that $J[S, X] = 0$ if and only if $v(X) = J(\nabla X)$. The local formula (2.16) follows by formulae (2.8) and (2.13). \square

Proposition 2.6. *Consider a semispray S , a nonlinear connection (h, v) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h, v))$. A vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ is a dynamical symmetry if and only if X is a newtonoid and satisfies*

$$(2.17) \quad \nabla(J\nabla X) + \Phi(X) = 0.$$

Proof. A vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ is a dynamical symmetry if and only if $h[S, X] = 0$ and $v[S, X] = 0$. Composing by J shows that the first condition is equivalent to $J[S, X] = 0$, which in turns is equivalent to $X \in \mathfrak{X}_S^1$. If $X \in \mathfrak{X}(TM \setminus \{0\})$ we have

$$v[S, X] = v[S, vX] + v[S, hX] = \nabla(vX) + \Phi(X).$$

Thus X is a dynamical symmetry if and only if $X \in \mathfrak{X}_S^1$ and $\nabla(vX) + \Phi(X) = 0$ and the result follows by Lemma 2.5. \square

We note that in the first term of formula (2.17), $J\nabla X$ only contain horizontal derivatives of X , while the outer ∇ acts on the vertical vector field $J\nabla X$. According to formula (2.7), the two actions might be different. The next proposition shows that the two actions of the dynamical covariant derivative ∇ coincide if and only if $\nabla J = 0$ is satisfied. Moreover, this condition is only satisfied for one nonlinear connection. In terms of S this unique nonlinear connection is explicitly given through its vertical projector $2v = \text{Id} + \mathcal{L}_S J$, which is the *canonical nonlinear connection* in [24]. The advantage of this method of assigning a nonlinear connection to a semispray is that the method also generalize to systems of higher order ODEs. This is the topic of Section 3.

Proposition 2.7. *Consider a semispray S , a nonlinear connection (h, v) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h, v))$. Then, the following conditions are equivalent:*

- i) ∇ restricts to a map $\nabla : \mathfrak{X}_S^1 \rightarrow \mathfrak{X}_S^1$ that satisfies the Leibniz rule with respect to the $*$ product;
- ii) $\nabla J = 0$;
- iii) $\mathcal{L}_S J + \text{Id} - 2v = 0$;
- iv) $N_j^i = \partial G^i / \partial y^j$.

Proof. The formulae in (2.10) show that conditions ii), iii) and iv) are equivalent.

To prove implication ii) \Rightarrow i), let $X \in \mathfrak{X}_S^1$. According to Lemma 2.5, we have that $vX = J\nabla X$. We apply ∇ to both sides of this equality, use $\nabla v = 0$ and $\nabla J = 0$, and obtain $v(\nabla X) = \nabla vX = \nabla(J\nabla X) = J\nabla(\nabla X)$, which according to Lemma 2.5 shows that $\nabla X \in \mathfrak{X}_S^1$.

Let $f \in C^\infty(TM \setminus \{0\})$ and $X \in \mathfrak{X}_S^1$. Since $f * X = fX + S(f)JX$ and ∇ satisfies the Leibniz rule for the \cdot product, it follows that

$$(2.18) \quad \begin{aligned} \nabla(f * X) &= S(f)X + f\nabla X + S^2(f)JX + S(f)\nabla(JX), \\ \nabla(f) * X + f * \nabla X &= S(f)X + f\nabla X + S^2(f)JX + S(f)J(\nabla X). \end{aligned}$$

Using the condition $\nabla J = 0$ we obtain $\nabla(f * X) = \nabla(f) * X + f * \nabla X$.

To prove implication $i) \Rightarrow ii)$, we will show that the $(1,1)$ -type tensor field ∇J vanishes on the set $\mathfrak{X}_S^1 \cup \mathfrak{X}^v(TM \setminus \{0\})$, which is a set of generators for $\mathfrak{X}(TM \setminus \{0\})$. By formulae (2.4) and (2.6), ∇J vanishes on $\mathfrak{X}^v(TM \setminus \{0\})$. Since ∇ satisfies the Leibniz rule with respect to the $*$ product, it follows from the two formulae (2.18) that $S(f)(\nabla J)X = 0$ for an arbitrary function $f \in C^\infty(TM \setminus \{0\})$ and arbitrary vector field $X \in \mathfrak{X}_S^1$. Therefore, $\nabla J = 0$ on \mathfrak{X}_S^1 , and this completes the proof of the proposition. \square

For a semispray S , we say that the nonlinear connection characterized by any of the four conditions of Proposition 2.7 is the *canonical nonlinear connection* of S . For the remaining of this section we will consider only this canonical nonlinear connection induced by a semispray S . The next proposition characterizes the dynamical and Lie symmetries of a semispray S using the dynamical covariant derivative ∇ associated to S and its canonical nonlinear connection.

Proposition 2.8. *Consider a semispray S .*

- i) *A vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ is a dynamical symmetry of S if and only if it is a newtonoid and*

$$(2.19) \quad \nabla^2 JX + \Phi(X) = 0.$$

- ii) *Locally, a newtonoid vector field $X \in \mathfrak{X}(TM \setminus \{0\})$ is a dynamical symmetry of S if and only if its horizontal components satisfy*

$$(2.20) \quad \nabla^2 X^i + R_j^i X^j = 0,$$

where ∇X^i is defined in equation (2.8).

- iii) *A vector field $X \in \mathfrak{X}(M)$ is a Lie symmetry of S if and only if*

$$(2.21) \quad \nabla^2 X^{1,0} + \Phi(X^{1,1}) = 0.$$

Proof. Since we are using the canonical nonlinear connection, we have that $\nabla J = 0$. The first claim follows using Proposition 2.6. For the second claim we use local formulae (2.8) and (2.3). Therefore, the local components of the vertical vector field $\nabla^2 JX + \Phi(X)$ are $\nabla^2 X^i + R_j^i X^j$. Hence formulae (2.19) and (2.20) are equivalent. For the third claim consider a vector field $X \in \mathfrak{X}(M)$. Its complete lift $X^{1,1}$ is a newtonoid for S and formula (2.21) follows from formula (2.19) since $JX^{1,1} = X^{1,0}$. \square

Equation (2.20) represents the covariant reformulation of equation (2.12), both terms $\nabla^2 X^i$ and $R_j^i X^j$ behave as components of a vector field from the base manifold.

2.3. First order variation and Jacobi fields for systems of SODE. Consider $c : I \rightarrow M$ a geodesic of a semispray S . A *geodesic variation* of c is a smooth map $V : I \times (-\varepsilon, \varepsilon) \rightarrow M$, $V = V(t, s)$ such that

- i) $V(t, 0) = c(t)$, for all t in I ,
- ii) $V(t, s)$ is a geodesic of S for all s in $(-\varepsilon, \varepsilon)$.

Definition 2.9. A vector field $\xi : I \rightarrow TM$, along a geodesic $c = \pi \circ \xi$ of a semispray S , is called a *Jacobi field* of S if ξ is the variation vector field, $\xi(t) = \partial_s V(t, s)|_{s=0}$, of some geodesic variation V .

Next we characterize the Jacobi fields of a semispray, and hence the geodesic variation, in terms of a system of differential equations, using the dynamical covariant derivative and the curvature components of the Jacobi endomorphism. We define first the dynamical covariant derivative of a vector field along a curve.

Consider $\xi : I \rightarrow TM$, $\xi(t) = (x^i(t), \xi^i(t))$, a vector field along a regular curve $c = \pi \circ \xi$. Then the *complete lift* of ξ is the vector field $\xi^{1,1} : I \rightarrow T(TM \setminus \{0\})$ along c' , defined as $\xi^{1,1}(t) = (x^i(t), dx^i/dt, \xi^i(t), d\xi^i/dt)$. The *vertical lift* of ξ is the vector field $\xi^{1,0} : I \rightarrow T(TM \setminus \{0\})$ along c' , defined as $\xi^{1,0}(t) = (x^i(t), dx^i/dt, 0, \xi^i(t))$. Since ξ is a vector field along c , for each $t \in I$, we can find a vector field $X \in \mathfrak{X}(M)$ such that $X \circ c = \xi$ near t . Then $\xi^{1,1} = X^{1,1} \circ c'$ and $\xi^{1,0} = X^{1,0} \circ c'$. Assume now that c is a geodesic, which means that c' is a geodesic of a semispray S . Then, we define

$$\nabla \xi^{1,0} = (\nabla X^{1,0}) \circ c'.$$

If locally $\xi = (x^i, \xi^i)$, then $\nabla \xi^{1,0} = \nabla \xi^i \partial / \partial y^i$, where

$$(2.22) \quad \nabla \xi^i = \frac{d\xi^i}{dt} + N_j^i \left(x, \frac{dx}{dt} \right) \xi^j.$$

Thus $\nabla \xi^{1,0}$ does not depend on the choice of X .

Proposition 2.10. *Consider a bounded interval I . A vector field $\xi : I \rightarrow TM$, along a geodesic $c = \pi \circ \xi$, is a Jacobi field if and only if*

$$(2.23) \quad \nabla^2 \xi^{1,0} + \Phi(\xi^{1,1}) = 0.$$

If locally $\xi(t) = (x^i(t), \xi^i(t))$, equation (2.23) is equivalent to

$$(2.24) \quad \nabla^2 \xi^i + R_j^i \left(x(t), \frac{dx}{dt} \right) \xi^j = 0.$$

Proof. Let $\xi : I \rightarrow TM$, $\xi(t) = (x^i(t), \xi^i(t))$ be a vector field along the geodesic $c = \pi \circ \xi$. Using formula (2.22), we have

$$(2.25) \quad \nabla^2 \xi^{1,0} + \Phi(\xi^{1,1}) = \left(\frac{d^2 \xi^i}{dt^2} + 2 \frac{\partial G^i}{\partial y^j} \left(x, \frac{dx}{dt} \right) \frac{d\xi^j}{dt} + 2 \frac{\partial G^i}{\partial x^j} \left(x, \frac{dx}{dt} \right) \xi^j \right) \frac{\partial}{\partial y^i} \Big|_{c'(t)}.$$

If ξ is a Jacobi field then there is a geodesic variation $V(t, s)$ such that $V(t, 0) = c(t)$,

$$\frac{\partial V^i}{\partial s}(t, 0) = \xi^i(t), \text{ and } \frac{\partial^2 V^i}{\partial t^2} + 2G^i \left(V, \frac{\partial V}{\partial t} \right) = 0, \forall (t, s) \in I \times (-\varepsilon, \varepsilon).$$

Differentiating the last equation with respect to s and setting $s = 0$ we obtain that the right hand side in formula (2.25) vanish and equations (2.23) and (2.24) follow.

Conversely, if $\xi : I \rightarrow TM$, $\xi(t) = (x^i(t), \xi^i(t))$ is a solution of the Jacobi equations (2.24) then right hand side in formula (2.25) vanish. Using the assumption that the interval I is bounded and [11, Theorem 4.4] it follows that ξ is a variation vector field for some geodesic variation. Therefore ξ is a Jacobi field. \square

Consider $c(t) = (x^i(t))$ a geodesic of a semispray S . Its tangent vector field $c'(t) = (x^i(t), dx^i/dt)$ is a Jacobi field of S . This can be seen either from the fact that the right hand side in formula (2.25) vanish for $\xi^i = dx^i/dt$ or from the fact that c' is the variation vector field of the geodesic variation $V(t, s) = c(t + s)$. In the Riemannian or the Finslerian context [4, 15] we can also see that the tangent vector field c' of a geodesic c satisfies the Jacobi equations (2.24) and hence it is a Jacobi field since the geodesic equations (2.2) can be written as $\nabla(dx^i/dt) = 0$ and the curvature components satisfy $R_j^i(x, dx/dt) dx^j/dt = 0$, [42, 43].

For a geodesic $c(t) = (x^i(t))$ of a semispray S , the vector field $(x^i(t), tdx^i/dt)$ is a geodesic of S if and only if $y^j \partial G^i / \partial y^j = 2G^i$. This condition means that functions G^i are 2-homogeneous in the fibre coordinates y . In this case S is called a *spray* and this includes Riemannian and Finslerian cases.

The second implication of Proposition 2.10 is proved in the Riemannian [15] or Finslerian context [4] using the exponential map. In the general context of a semispray, due to the lack of homogeneity, this exponential map cannot be defined and the proof in [11, Theorem 4.4] uses the flow of the complete lift of a semispray.

In the covariant form of the Jacobi equations (2.24), the components R_j^i of the Jacobi endomorphism contain information about the geodesic behavior and the stability of the geodesics of a semispray S . If the real part of the eigenvalues of the curvature components R_j^i are negative then the geodesics will spread out and will be unstable, while if they are positive the geodesics will be stable [39].

3. SYMMETRIES AND FIRST ORDER VARIATION FOR SYSTEMS OF HIGHER DIFFERENTIAL EQUATIONS

In this section we will extend the geometric theory for systems of SODE, developed in the previous sections, to systems of HODE. A system of $(k+1)$ order ordinary differential equations can be associated with a semispray S of order k , which is a special vector field on the k -th order tangent bundle. For a semispray of order k there are various nonlinear connections one can associate to it [1, 9, 12, 16, 18, 20]. In the first part of the section we consider a semispray of order k and an arbitrary nonlinear connection. For this pair we define the dynamical covariant derivatives of first and higher order, and the Jacobi endomorphism. The main contribution is to describe a particular nonlinear connection that yields covariant expressions for symmetries and first order variation of the system of HODE. For this nonlinear connection we explicitly compute all components of the Jacobi endomorphism and use these components to provide a geometric expression for the Wuenschmann invariant [20, 21, 38].

3.1. Systems of HODE and geometric structures on $T^k M$. The framework for studying systems of $(k+1)$ order ordinary differential equations on a manifold M is the tangent bundle $T^k M = J_0^k M$ of order $k \geq 1$ [18, 30, 34, 44]. This is the k -th order jet bundle of curves c from a neighborhood of $0 \in \mathbb{R}$ to M . For a curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, denote by $j^k c : I \rightarrow T^k M$, its k -th jet lift, defined as

$$j^k c(t) = \left(x^i(t), \frac{1}{1!} \frac{dx^i}{dt}(t), \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k}(t) \right).$$

If c is a regular curve, then $j^k c(t) \in T^k M \setminus \{0\}$ for all $t \in I$. Local coordinates on M are denoted by (x^i) and the induced local coordinates on $T^k M$ are denoted by $(x^i, y^{(1)i}, \dots, y^{(k)i})$, where

$$y^{(\alpha)i}(j_0^k c) = \frac{1}{\alpha!} \frac{d^\alpha (x^i(c(t)))}{dt^\alpha} \Big|_{t=0}, \quad \alpha \in \{1, \dots, k\}.$$

Let also $y^{(0)i} = x^i$. Denote $T^0 M = M$. For each $\alpha \in \{0, 1, \dots, k-1\}$, the canonical submersion $\pi_\alpha^k : T^k M \rightarrow T^\alpha M$ induces a natural foliation of $T^k M$. Consider also the subbundle $T^k M \setminus \{0\} = \{(x, y^{(1)}, \dots, y^{(k)}) \in T^k M, y^{(1)} \neq 0\}$. Note that $T^k M \setminus \{0\} = (\pi_1^k)^{-1}(TM \setminus \{0\})$.

The *tangent structure* (or vertical endomorphism) of order k is the $(1,1)$ -type tensor field on $T^k M$ defined as

$$(3.1) \quad J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}.$$

It has the following properties: $J^{k+1} = 0$, $\text{Im } J^\alpha = \text{Ker } J^{k-\alpha+1}$, $\alpha \in \{1, \dots, k\}$.

The foliated structure of $T^k M$ determines k regular vertical distributions

$$V_\alpha(u) = \text{Ker } D_u \pi_{\alpha-1}^k = \text{Im } J_u^\alpha = \text{Ker } J_u^{k-\alpha+1}, \text{ for } u \in T^k M, \alpha \in \{1, \dots, k\}.$$

For each $\alpha \in \{1, \dots, k\}$, the distribution V_α is tangent to the fibers of $\pi_{\alpha-1}^k : (x^i, y^{(1)i}, \dots, y^{(k)i}) \rightarrow (x^i, y^{(1)i}, \dots, y^{(\alpha-1)i})$, and hence it is integrable. Note that $\dim V_\alpha = (k - \alpha + 1)n$, $\alpha \in \{1, \dots, k\}$ and $V_k(u) \subset V_{k-1}(u) \subset \dots \subset V_1(u)$, for each $u \in T^k M$.

In this section we pay attention to the geometry of a system of $(k+1)$ order ordinary differential equations. Such a system of HODE on M can be represented using a semispray of order k , which is a special vector field on $T^k M \setminus \{0\}$.

A *semispray of order k* on M is a vector field $S \in \mathfrak{X}(T^k M \setminus \{0\})$ such that any integral curve of S , $\gamma : I \rightarrow T^k M \setminus \{0\}$ is of the form $\gamma = j^k(\pi_0^k \circ \gamma)$. In induced coordinates for $T^k M \setminus \{0\}$, a semispray of order k is given by

$$(3.2) \quad S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i \frac{\partial}{\partial y^{(k)i}},$$

for some functions G^i defined on domains of induced local charts. For an integral curve $\gamma : I \rightarrow T^k M \setminus \{0\}$ of S , we say that curve $c = \pi_0^k \circ \gamma$ is a *geodesic* of S . Therefore, a regular curve $c : I \rightarrow M$ is a geodesic of S if and only if $S \circ j^k c = (j^k c)'$. Locally, a regular curve $c : I \rightarrow M$, $c(t) = (x^i(t))$, is a geodesic of S if and only if it satisfies the system of $(k+1)$ order ordinary differential equations

$$(3.3) \quad \frac{1}{(k+1)!} \frac{d^{k+1} x^i}{dt^{k+1}} + G^i \left(x, \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) = 0.$$

Thus a semispray of order k describes systems of HODE with regular curves on M as solutions.

Following the line developed in Section 2 we will show that the geometry of a semispray of order k can be determined from a canonical nonlinear connection, associated to the semispray. An n -dimensional distribution $H_0 : u \in T^k M \setminus \{0\} \mapsto H_0(u) \subset T_u T^k M$, that is supplementary to the vertical distribution $V_1(u)$, is called a *nonlinear connection*, or *horizontal distribution*. Therefore, for a nonlinear connection, we have the following decomposition $T_u T^k M = H_0(u) \oplus V_1(u)$, for all $u \in T^k M \setminus \{0\}$. For a nonlinear connection H_0 , consider also the n -dimensional distributions $H_\alpha = J^\alpha(H_0)$, for $\alpha \in \{1, \dots, k-1\}$. Therefore, for a nonlinear connection, we have the following decomposition:

$$(3.4) \quad T_u T^k M = \oplus_{\alpha=0}^{k-1} H_\alpha(u) \oplus V_k(u), \forall u \in T^k M \setminus \{0\}.$$

The set of k distributions H_0, H_1, \dots, H_{k-1} is called a multi-connection in [41]. We will denote by $h_0, h_1, \dots, h_{k-1}, v_k$ the projectors that correspond to decomposition (3.4), which characterizes the nonlinear connection. Sometimes, for uniform notations, we also denote $h_k = v_k$. Decomposition (3.4) induces also a decomposition of the vertical subspaces

$$(3.5) \quad V_\alpha(u) = \oplus_{\beta=\alpha}^{k-1} H_\beta(u) \oplus V_k(u), \forall u \in T^k M \setminus \{0\}, \forall \alpha \in \{1, \dots, k-1\}.$$

Decomposition (3.5) implies that $\text{Id}|_{\text{Im } J^\alpha} = \text{Id}|_{\text{Ker } J^{k-\alpha+1}} = \sum_{\beta=\alpha}^k h_\beta$, for all $\alpha \in \{0, \dots, k\}$. Therefore, for all $\alpha \in \{0, \dots, k\}$ we have the following identities

$$(3.6) \quad J^\alpha = \sum_{\beta=\alpha}^k h_\beta \circ J^\alpha, \quad J^{k-\alpha+1} = \sum_{\beta=0}^{\alpha-1} J^{k-\alpha+1} \circ h_\beta.$$

For $\alpha, \beta \in \{0, \dots, k\}$, one can reformulate the above identities as follows

$$(3.7) \quad h_\beta \circ J^\alpha = \begin{cases} 0, & \text{if } \beta \leq \alpha - 1, \\ J^\alpha \circ h_{\beta-\alpha}, & \text{if } \beta \geq \alpha, \end{cases}$$

$$(3.8) \quad J^\alpha \circ h_\beta = \begin{cases} 0, & \text{if } \alpha + \beta \geq k + 1, \\ h_{\alpha+\beta} \circ J^\alpha, & \text{if } \alpha + \beta \leq k. \end{cases}$$

For $\alpha \in \{0, \dots, k-1\}$, we denote by $\mathfrak{X}^{h_\alpha}(T^k M \setminus \{0\}) = h_\alpha(\mathfrak{X}(T^k M \setminus \{0\}))$, $\mathfrak{X}^{v_k}(T^k M \setminus \{0\}) = v_k(\mathfrak{X}(T^k M \setminus \{0\}))$, the $C^\infty(T^k M \setminus \{0\})$ -modules of horizontal and respectively vertical vector fields.

For a semispray of order k , S , and a nonlinear connection (h_α, v_k) , consider the vertically valued, $(1, 1)$ -type tensor field on $T^k M \setminus \{0\}$

$$(3.9) \quad \Phi = -v_k \circ \mathcal{L}_S v_k = v_k \circ \sum_{\alpha=0}^{k-1} \mathcal{L}_S h_\alpha = v_k \circ \sum_{\alpha=0}^{k-1} (\mathcal{L}_S \circ h_\alpha)$$

that we will call the *Jacobi endomorphism*. Here, for a $(1, 1)$ -type tensor field A on $T^k M \setminus \{0\}$, $\mathcal{L}_S A = \mathcal{L}_S \circ A - A \circ \mathcal{L}_S$ denotes the Frölicher-Nijenhuis bracket of S and A . We will obtain now the local expression of the Jacobi endomorphism. On the domain of induced local charts on $T^k M \setminus \{0\}$, consider the vector fields

$$(3.10) \quad \frac{\delta}{\delta x^i} = h_0 \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} - \sum_{\beta=1}^k N_{(\beta)i}^j \frac{\partial}{\partial y^{(\beta)j}}, \quad \frac{\delta}{\delta y^{(\alpha)i}} = J^\alpha \left(\frac{\delta}{\delta x^i} \right),$$

for $\alpha \in \{1, \dots, k\}$, which form a basis for $\mathfrak{X}(T^k M \setminus \{0\})$, adapted to the decomposition (3.4). Functions $N_{(\alpha)j}^i$ are locally defined on $T^k M \setminus \{0\}$ and are called the coefficients of the nonlinear connection. The dual basis to the basis (3.10) is given by the following locally defined 1-forms on $T^k M \setminus \{0\}$:

$$(3.11) \quad dx^i, \delta y^{(\alpha)i} = dy^{(\alpha)i} + \sum_{\beta=1}^{\alpha} M_{(\beta)j}^i dy^{(\alpha-\beta)j}, \quad \alpha \in \{1, \dots, k\},$$

where the dual coefficients $M_{(\beta)j}^i$ are given by [34]

$$(3.12) \quad M_{(1)j}^i = N_{(1)j}^i, \quad M_{(\alpha)j}^i = N_{(\alpha)j}^i + \sum_{\beta=1}^{\alpha-1} N_{(\alpha-\beta)s}^i M_{(\beta)j}^s, \quad \alpha \in \{2, \dots, k\}.$$

A change of induced local coordinates $(x^i, y^{(\alpha)i}) \rightarrow (\tilde{x}^i(x), \tilde{y}^{(\alpha)i}(x, y^{(\beta)})$) on $T^k M$ induces the following transformation rule for the basis (3.10) and its dual basis (3.11)

$$\begin{aligned} \frac{\delta}{\delta x^j} &= \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\delta}{\delta \tilde{x}^i}, \quad \frac{\delta}{\delta y^{(\alpha)j}} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\delta}{\delta \tilde{y}^{(\alpha)i}}, \quad \frac{\partial}{\partial y^{(k)j}} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial}{\partial \tilde{y}^{(k)i}}, \quad \alpha \in \{1, \dots, k-1\} \\ dx^i &= \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \quad \delta \tilde{y}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} \delta y^{(\alpha)j}, \quad \alpha \in \{1, \dots, k\}. \end{aligned}$$

Therefore the components of a tensor field on $T^k M$, with respect to these bases, will transform as the components of a tensor field on the base manifold M .

The projectors (h_α, v_k) that correspond to the decomposition (3.4) can be expressed, with respect to the bases (3.10) and (3.11) as follows

$$h_0 = \frac{\delta}{\delta x^i} \otimes dx^i, \quad h_\alpha = \frac{\delta}{\delta y^{(\alpha)i}} \otimes \delta y^{(\alpha)i}, \quad \alpha \in \{1, \dots, k-1\}, \quad v_k = \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(k)i}.$$

With respect to the bases (3.10) and (3.11) the tangent structure of order k has the following expression:

$$(3.13) \quad J = \frac{\delta}{\delta y^{(1)i}} \otimes dx^i + \frac{\delta}{\delta y^{(2)i}} \otimes \delta y^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(k-1)i}.$$

Locally, the Jacobi endomorphism can be expressed as follows

$$(3.14) \quad \Phi = \sum_{\alpha=0}^{k-1} R_{(\alpha)j}^i \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(\alpha)j}.$$

Under a change of induced local coordinates on $T^k M$, the k components $R_{(\alpha)j}^i$ of the Jacobi endomorphism transform as the components of a $(1,1)$ -type tensor field on the base manifold. We will refer to $R_{(\alpha)j}^i$ as to the *curvature components*. In Subsection 4.1 we will show that in the Riemannian context, these components are functions of the Riemannian curvature and its dynamical covariant derivatives.

We will provide now explicit formulae for the curvature components $R_{(\alpha)j}^i$ of the Jacobi endomorphism. For a semispray of order k , S , and a nonlinear connection with coefficients $N_{(\alpha)j}^i$ we introduce the following notations:

$$(3.15) \quad \begin{aligned} I_{(2)j}^i &= 2N_{(2)j}^i - S(N_{(1)j}^i) + N_{(1)j}^s N_{(1)s}^i, \\ I_{(\alpha)j}^i &= \alpha N_{(\alpha)j}^i - S(N_{(\alpha-1)j}^i) + N_{(1)j}^s N_{(\alpha-1)s}^i + \sum_{\beta=2}^{\alpha-1} I_{(\beta)j}^s N_{(\alpha-\beta)s}^i, \end{aligned}$$

for $\alpha \in \{3, \dots, k\}$. With respect to these notations we have the following formulae for the Lie bracket of the semispray S and the vectors of the basis (3.10)

$$(3.16) \quad \begin{aligned} \left[S, \frac{\delta}{\delta x^j} \right] &= N_{(1)j}^i \frac{\delta}{\delta x^i} + \sum_{\beta=2}^k I_{(\beta)j}^i \frac{\delta}{\delta y^{(\beta-1)i}} \\ &+ \left\{ (k+1) \frac{\delta G^i}{\delta x^j} - S(N_{(k)j}^i) + N_{(1)j}^s N_{(k)s}^i + \sum_{\beta=2}^k I_{(\beta)j}^s N_{(k+1-\beta)s}^i \right\} \frac{\partial}{\partial y^{(k)i}}, \\ \left[S, \frac{\delta}{\delta y^{(\alpha)j}} \right] &= -\alpha \frac{\delta}{\delta y^{(\alpha-1)j}} + N_{(1)j}^i \frac{\delta}{\delta y^{(\alpha)i}} + \sum_{\beta=2}^{k-\alpha} I_{(\beta)j}^i \frac{\delta}{\delta y^{(\alpha+\beta-1)i}} \\ &+ \left\{ (k+1) \frac{\delta G^i}{\delta y^{(\alpha)j}} - \alpha N_{(k+1-\alpha)j}^i - S(N_{(k-\alpha)j}^i) + N_{(1)j}^s N_{(k-\alpha)s}^i \right. \\ &\quad \left. + \sum_{\beta=2}^{k-\alpha} I_{(\beta)j}^s N_{(k+1-\alpha-\beta)s}^i \right\} \frac{\partial}{\partial y^{(k)i}}, \end{aligned}$$

for each $\alpha \in \{1, \dots, k-1\}$. From formulae (3.16) it follows that $\delta y^{(\alpha)i} [S, \delta / \delta y^{(\beta)j}]$ behave as $(1,1)$ -type tensors on M , for all $\alpha \neq \beta \in \{0, 1, \dots, k\}$. Therefore, $I_{(\alpha)j}^i$, for $\alpha \in \{2, \dots, k\}$, behave as $(1,1)$ -type tensors on M .

From formulae (3.9) and (3.14) it follows that the curvature components of the Jacobi endomorphism can be computed as follows

$$R_{(\alpha)j}^i \frac{\partial}{\partial y^{(k)i}} = v_k \left[S, \frac{\delta}{\delta y^{(\alpha)j}} \right], \quad \alpha \in \{0, 1, \dots, k-1\}.$$

Using the expressions (3.16) we obtain the following formulae for $R_{(\alpha)j}^i$, for $\alpha \in \{0, 1, \dots, k-1\}$

$$(3.17) \quad \begin{aligned} R_{(\alpha)j}^i &= (k+1) \frac{\delta G^i}{\delta y^{(\alpha)j}} - \alpha N_{(k+1-\alpha)j}^i - S(N_{(k-\alpha)j}^i) + N_{(1)j}^s N_{(k-\alpha)s}^i \\ &+ \sum_{\beta=2}^{k-\alpha} I_{(\beta)j}^s N_{(k+1-\alpha-\beta)s}^i. \end{aligned}$$

For a k -semispray S and a nonlinear connection (h_α, v_k) , consider the map $\nabla : \mathfrak{X}(T^k M \setminus \{0\}) \rightarrow \mathfrak{X}(T^k M \setminus \{0\})$, given by

$$(3.18) \quad \nabla = \sum_{\alpha=0}^{k-1} h_\alpha \circ \mathcal{L}_S \circ h_\alpha + v_k \circ \mathcal{L}_S \circ v_k = \mathcal{L}_S + \sum_{\alpha=0}^{k-1} h_\alpha \circ \mathcal{L}_S h_\alpha + v_k \circ \mathcal{L}_S v_k.$$

We call ∇ the *dynamical covariant derivative* associated to the pair $(S, (h_\alpha, v_k))$. By setting $\nabla f = S(f)$, for $f \in C^\infty(T^k M \setminus \{0\})$, using the Leibniz rule, and the requirement that ∇ commutes with tensor contraction, we extend the action of ∇ to arbitrary tensor fields and forms on $T^k M \setminus \{0\}$. Formula (3.18) implies that $\nabla h_\alpha = 0$ for every $\alpha \in \{0, \dots, k-1\}$. Hence ∇ preserves all distributions H_α . Similarly, we obtain that $\nabla v_k = 0$, hence it preserves the vertical distribution V_k . From formula (3.5) it follows that ∇ preserves also all the vertical distributions V_α . Dynamical covariant derivative ∇ acts in a similar way on the distributions H_α , but has a different action on the vertical distribution V_k . This can be seen locally as follows. Using formulae (3.16) and (3.18) we obtain

$$(3.19) \quad \begin{aligned} \nabla \frac{\delta}{\delta y^{(\alpha)i}} &= h_\alpha \left[S, \frac{\delta}{\delta y^{(\alpha)i}} \right] = N_{(1)i}^j \frac{\delta}{\delta y^{(\alpha)j}}, \quad \alpha \in \{0, \dots, k-1\} \\ \nabla \frac{\partial}{\partial y^{(k)i}} &= v_k \left[S, \frac{\partial}{\partial y^{(k)i}} \right] = \left((k+1) \frac{\partial G^j}{\partial y^{(k)i}} - k N_{(1)i}^j \right) \frac{\partial}{\partial y^{(k)j}}. \end{aligned}$$

The action of the dynamical covariant derivative ∇ on a horizontal vector field $X = h_\alpha X$, for $\alpha \in \{0, 1, \dots, k-1\}$, is given by

$$(3.20) \quad \nabla X = \nabla \left(X^i \frac{\delta}{\delta y^{(\alpha)i}} \right) = \nabla X^i \frac{\delta}{\delta y^{(\alpha)i}}, \quad \nabla X^i = S(X^i) + N_{(1)j}^i X^j.$$

The first order differential operator ∇X^i and its higher order iterations were considered by Kosambi [28]. For $k = 2$, iterated actions of ∇ were also considered in [9] in order to fix a nonlinear connection for a second order Lagrange space.

Next lemma gives some compatibility conditions between the geometric structures introduced so far. In this lemma we do not assume any relation between the semispray and the nonlinear connection. However, we will use this lemma to fix partially the nonlinear connection later on.

Lemma 3.1. *Consider a semispray S , a nonlinear connection (h_α, v_k) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h_\alpha, v_k))$. Consider $\alpha, \beta \in \{0, \dots, k\}$. Then, the following formulae are true:*

$$(3.21) \quad J \circ \mathcal{L}_S J^\alpha = -\alpha J^\alpha,$$

$$(3.22) \quad h_\beta \circ \mathcal{L}_S \circ J^\alpha = \begin{cases} -(\beta+1)h_\beta \circ J^\beta, & \text{if } \beta = \alpha - 1 \\ 0, & \text{if } \beta < \alpha - 1, \end{cases}$$

$$(3.23) \quad J^\beta \circ \mathcal{L}_S \circ h_\alpha = \begin{cases} -\alpha J^{k-\alpha} \circ h_\alpha, & \text{if } \alpha = k - \beta + 1 \\ 0, & \text{if } \alpha > k - \beta + 1, \end{cases}$$

$$(3.24) \quad \nabla J^\alpha \circ J^{k-\alpha} = \nabla J^k = \mathcal{L}_S J^k + (k h_{k-1} - v_k) \circ J^{k-1}, \quad \text{if } \alpha \geq 1,$$

$$(3.25) \quad \nabla J^\alpha = (k+1) \left(\frac{\partial G^i}{\partial y^{(k)j}} - N_{(1)j}^i \right) \frac{\partial}{\partial y^{(k)i}} \otimes \delta y^{(k-\alpha)j}, \text{ if } \alpha \geq 1,$$

$$(3.26) \quad \begin{aligned} \mathcal{L}_S J &+ \text{Id} - (k+1)v_k = \\ \nabla J &+ \sum_{\alpha=1}^{k-1} (k+1) \left(\frac{\delta G^j}{\delta y^{(\alpha)i}} - N_{(k+1-\alpha)i}^j \right) \frac{\partial}{\partial y^{(k)j}} \otimes \delta y^{(\alpha-1)i}. \end{aligned}$$

Proof. From formula (3.1), a local computation shows that for every $\alpha \in \{1, \dots, k\}$ and for every $X \in \mathfrak{X}(T^k M \setminus \{0\})$ we have

$$(3.27) \quad [S, J^\alpha X] - J^\alpha [S, X] + \alpha J^{\alpha-1} X \in \text{Ker } J = \text{Im } J^k.$$

For $\alpha \in \{1, \dots, k\}$, formula (3.21) follows by composing in formula (3.27) to the left with J . For $\alpha = 0$, formula (3.21) follows since $\mathcal{L}_S \text{Id} = 0$.

Formulae (3.22) follow from formula (3.27) by composing to the left with h_β and using the identities (3.7).

Formulae (3.23) follow from formula (3.27) by composing to the right with h_β and using the identities (3.8).

From formulae (3.18), (3.7), (3.8), (3.22), and (3.23) for $\beta = k$, we obtain

$$\begin{aligned} \nabla \circ J^k &= v_k \circ \mathcal{L}_S \circ J^k = \mathcal{L}_S \circ J^k - (h_0 + \dots + h_{k-1}) \circ \mathcal{L}_S \circ J^k \\ &= \mathcal{L}_S \circ J^k + k h_{k-1} \circ J^{k-1}, \\ J^k \circ \nabla &= J^k \circ \mathcal{L}_S \circ h_0 = J^k \circ \mathcal{L}_S - J^k \circ \mathcal{L}_S \circ (h_1 + \dots + h_{k-1} + v_k) \\ &= J^k \circ \mathcal{L}_S + J^{k-1} \circ h_1 = J^k \circ \mathcal{L}_S + v_k \circ J^{k-1}. \end{aligned}$$

Now the second equality in formula (3.24) follows using the fact that $\nabla J^k = \nabla \circ J^k - J^k \circ \nabla$.

We will prove now the first equality of formula (3.24). Since $\nabla J^\alpha \circ J^{k-\alpha} = \nabla \circ J^k - J^\alpha \circ \nabla \circ J^{k-\alpha}$, it remains to show that $J^\alpha \circ \nabla \circ J^{k-\alpha} = J^k \circ \nabla$, for $\alpha \in \{1, \dots, k\}$. Using formula (3.18) we have

$$J^\alpha \circ \nabla \circ J^{k-\alpha} = J^\alpha \circ h_{k-\alpha} \circ \mathcal{L}_S \circ h_{k-\alpha} \circ J^{k-\alpha} = v_k \circ J^\alpha \circ \mathcal{L}_S \circ J^{k-\alpha} \circ h_0.$$

In formula (3.21), we replace α by $k - \alpha$, compose to the left with $J^{\alpha-1}$ and obtain

$$J^\alpha \circ \mathcal{L}_S \circ J^{k-\alpha} = J^k \circ \mathcal{L}_S - (k - \alpha) J^{k-1}.$$

Using the above two formulae and the fact that $v_k \circ J^{k-1} \circ h_0 = 0$, we obtain

$$J^\alpha \circ \nabla \circ J^{k-\alpha} = J^k \circ \mathcal{L}_S \circ h_0 = J^k \circ \nabla.$$

Formula (3.25) follows from expression (3.13) and the local expression of the dynamical covariant derivative (3.19).

By direct calculation it follows that for $\alpha \in \{0, \dots, k-1\}$

$$\begin{aligned} (\text{Id} + \mathcal{L}_S J) \left(\frac{\delta}{\delta y^{(\alpha)i}} \right) &= (k+1) \left(\frac{\delta G^j}{\delta y^{(\alpha+1)i}} - N_{(k-\alpha)i}^j \right) \frac{\partial}{\partial y^{(k)j}}, \\ (\mathcal{L}_S J) \left(\frac{\partial}{\partial y^{(k)i}} \right) &= k \frac{\partial}{\partial y^{(k)i}}. \end{aligned}$$

Using these formulae and (3.25) for $\alpha = 1$ it follows that formula (3.26) is true. \square

When $\beta = 1$ formulae (3.22) and (3.23) read as follows

$$(3.28) \quad h_0 \circ \mathcal{L}_S \circ J = -h_0, \quad J \circ \mathcal{L}_S \circ v_k = -k v_k.$$

Formula (3.26) generalizes formula (2.10). In the case $k = 1$ the tensor ∇J uniquely determines the tensor $\mathcal{L}_S J + \text{Id} - 2v$ that fixes the canonical nonlinear connection. For the case $k > 1$, a nonlinear

connection can be fixed using the (1,1)-type tensor $\mathcal{L}_S J + \text{Id} - (k+1)v_k$. Using formula (3.26), in Theorem 3.10, we will determine this tensor by fixing ∇J and

$$(3.29) \quad \sum_{\alpha=1}^{k-1} (k+1) \left(\frac{\delta G^j}{\delta y^{(\alpha)i}} - N_{(k+1-\alpha)i}^j \right) \frac{\partial}{\partial y^{(k)j}} \otimes \delta y^{(\alpha-1)i}.$$

Therefore, in Theorem 3.10, we will determine a nonlinear connection for a semispray S by fixing ∇J and the (1,1)-type tensor whose local expression is given in formula (3.29).

3.2. Symmetries for systems of HODE. For a semispray of order k there are various nonlinear connections one can associate to it [1, 6, 16, 18, 36]. In this section we will show what are the advantages of using the nonlinear connection proposed by Miron and Atanasiu [36]. We show that this connection is very useful for studying the symmetries and the first order variation of a system of higher order ordinary differential equations. Moreover, we show that with respect to this nonlinear connection one can explicitly write down all components of the Jacobi endomorphism. We use these components to provide a geometric expression for the Wuenschmann invariant [20, 21, 38] as well as for an invariant introduced by Fels [22] for the inverse problem of a fourth order ODE.

Definition 3.2. A vector field $X \in \mathfrak{X}(T^k M \setminus \{0\})$ is a *dynamical symmetry* of a k -semispray S if $[S, X] = 0$.

A direct calculation shows that a vector field on $T^k M \setminus \{0\}$,

$$(3.30) \quad X = X^i \left(x, y^{(\beta)} \right) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^k Y^{(\alpha)i} \left(x, y^{(\beta)} \right) \frac{\partial}{\partial y^{(\alpha)i}},$$

is a dynamical symmetry if and only if its components satisfy $\alpha! Y^{(\alpha)i} = S^\alpha(X^i)$, for all $\alpha \in \{1, \dots, k\}$ and

$$(3.31) \quad S^{k+1}(X^i) + (k+1)!X(G^i) = 0.$$

It follows that, while studying dynamical symmetries for a k -semispray S , the following set of vector fields on $T^k M \setminus \{0\}$ plays an important role:

$$(3.32) \quad \mathfrak{X}_S^k = \left\{ X \in \mathfrak{X}(T^k M \setminus \{0\}), X = X^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^k \frac{1}{\alpha!} S^\alpha(X^i) \frac{\partial}{\partial y^{(\alpha)i}} \right\}.$$

A vector field $X \in \mathfrak{X}_S^k$ is called a *newtonoid* of order k . Next, we will provide global characterizations, in terms of some differential operators for the set of newtonoid vector fields \mathfrak{X}_S^k and their images $J^\alpha(\mathfrak{X}_S^k)$, for $\alpha \in \{1, \dots, k\}$.

Consider $\pi_S^k : \mathfrak{X}(T^k M \setminus \{0\}) \rightarrow \mathfrak{X}(T^k M \setminus \{0\})$ the differential operator of order k defined as

$$\pi_S^k = \text{Id} + \frac{1}{1!} J \circ \mathcal{L}_S + \dots + \frac{1}{k!} J^k \circ \mathcal{L}_S^k.$$

The form of the differential operator π_S^k was inspired by the generalized Cartan operator acting on 1-forms on $T^k M$, considered in [18, 40].

Lemma 3.3. For each $\alpha \in \{0, \dots, k\}$, the set $J^\alpha(\mathfrak{X}_S^k)$ can be expressed, without local coordinates, as follows

$$(3.33) \quad J^\alpha(\mathfrak{X}_S^k) = \text{Ker}(J \circ \mathcal{L}_S + \alpha \text{Id}) \cap \text{Im } J^\alpha = \text{Im}(J^\alpha \circ \pi_S^k).$$

Proof. First, we prove the two equalities in formula (3.33) for $\alpha = 0$. Consider a vector field X on $T^k M \setminus \{0\}$, given locally by formula (3.30). It follows that $J[S, X] = 0$ if and only if its components satisfy $\beta!Y^{(\beta)i} = S^\beta(X^i)$, for all $\beta \in \{1, \dots, k\}$. This proves first equality in formula (3.33).

From formula (3.21) we have $J \circ \mathcal{L}_S \circ J = J^2 \circ \mathcal{L}_S - J$. Using this formula recurrently, we obtain that $(J \circ \mathcal{L}_S) \circ \pi_S^k = 0$. Therefore, $\text{Im } \pi_S^k \subset \text{Ker } (J \circ \mathcal{L}_S)$.

For the converse inclusion, consider a vector field X on $T^k M \setminus \{0\}$ such that $J[S, X] = 0$. It follows that $\mathcal{L}_S^\beta X \in \text{Ker } J^\beta$, for all $\beta \in \{1, \dots, k\}$. Hence, $(J^\beta \circ \mathcal{L}_S^\beta) X = 0$ and $X = \pi_S^k(X)$.

Consider now $\alpha \in \{1, \dots, k\}$. Using the above considerations it follows that $J^\alpha(\mathfrak{X}_S^k) = \text{Im}(J^\alpha \circ \pi_S^k)$. We prove now the first equality in formula (3.33). Consider $Z = J^\alpha X$ for $X \in \mathfrak{X}_S^k$. Therefore $J[S, X] = 0$ and using formula (3.21) we have $J[S, J^\alpha X] + \alpha J^\alpha X = J^{\alpha+1}[S, X] = 0$. It follows that $Z \in \text{Ker } (J \circ \mathcal{L}_S + \alpha \text{Id}) \cap \text{Im } J^\alpha$.

For the converse inclusion, consider $Z = J^\alpha X$ for $X \in \mathfrak{X}(T^k M \setminus \{0\})$ such that $J[S, Z] + \alpha Z = 0$. Using again formula (3.21) it follows that $J^{\alpha+1}[S, X] = 0$. If X is given locally by formula (3.30), then its components satisfy $\beta!Y^{(\beta)i} = S^\beta(X^i)$, for all $\beta \in \{1, \dots, k - \alpha\}$. It follows that $Z = J^\alpha \pi_S^k X \in J^\alpha(\mathfrak{X}_S^k)$. \square

For a vector field $X = X^i \partial / \partial x^i \in \mathfrak{X}(M)$, the *complete lift*, $X^{k,k} \in \mathfrak{X}(T^k M \setminus \{0\})$, is the vector field defined as

$$(3.34) \quad X^{k,k} = X^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^k \frac{1}{\alpha!} d_T^\alpha(X^i(x)) \frac{\partial}{\partial y^{(\alpha)i}},$$

where d_T is the *Tulczyjev operator* [44]

$$d_T = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.$$

For each $\alpha \in \{0, \dots, k-1\}$, the *vertical lift* of a vector field $X \in \mathfrak{X}(M)$ is the vector field $X^{k,\alpha} = J^{k-\alpha}(X^{k,k}) \in \mathfrak{X}(T^k M \setminus \{0\})$, [18, 29]. We denote by $\mathfrak{X}^{k,k}(T^k M \setminus \{0\})$, the set of complete lifts of vector fields on M and by $\mathfrak{X}^{k,\alpha}(T^k M \setminus \{0\})$, for $\alpha \in \{0, \dots, k-1\}$, the set of vertical lifts of vector fields on M . From expressions (3.32) and (3.34) it follows that complete lifts, vertical lifts, and newtonoid vector fields are related by

$$(3.35) \quad \mathfrak{X}^{k,\alpha}(T^k M \setminus \{0\}) \subseteq \bigcap_{S \text{ semispray}} J^{k-\alpha}(\mathfrak{X}_S^k),$$

for all $\alpha \in \{0, \dots, k\}$, and the equality holds if and only if $\dim M \geq 2$ and $\alpha = k$. If $\dim M = 1$ the above inclusion is strict due to the fact that the fibers of $T^k M \setminus \{0\}$ are not connected.

Definition 3.4. A vector field $X \in \mathfrak{X}(M)$ is a *Lie symmetry*, or a *point symmetry*, of a k -semispray S if the complete lift $X^{k,k}$ is a dynamical symmetry, which means $[S, X^{k,k}] = 0$.

Next, we define a C^∞ -module structure on the set of newtonoid vector fields of order k . This structure is inspired by a module structure on a set of 1-forms associated to adjoint symmetries of a system of HODE, [40]. In proposition 3.8 we show that this structure can be used to fix a part of a canonical nonlinear connection one can associate to a semispray.

Remark 3.5. For $f \in C^\infty(T^k M \setminus \{0\})$ and $X \in \mathfrak{X}_S^k$, we define

$$(3.36) \quad f * X = \pi_S^k(fX) = \sum_{\alpha=0}^k \frac{1}{\alpha!} S^\alpha(f) J^\alpha X.$$

- i) With respect to the product $*$, the set $J^\alpha(\mathfrak{X}_S^k)$ has the structure of a $C^\infty(T^k M \setminus \{0\})$ -module and the set $\mathfrak{X}^{k,\alpha}(T^k M \setminus \{0\})$ has the structure of a $C^\infty(M)$ -module, for all $\alpha \in \{0, 1, \dots, k\}$.
- ii) The map $h_\alpha : (J^\alpha(\mathfrak{X}_S^k), *) \rightarrow (\mathfrak{X}^{h_\alpha}(T^k M \setminus \{0\}), \cdot)$ is an isomorphism of $C^\infty(T^k M \setminus \{0\})$ -modules, for all $\alpha \in \{0, \dots, k\}$. The map $D\pi_0^k : (\mathfrak{X}^{k,k}(T^k M \setminus \{0\}), *) \rightarrow (\mathfrak{X}(M), \cdot)$ is an isomorphism of $C^\infty(M)$ -modules.
- iii) A vector field X on $T^k M \setminus \{0\}$ belongs to the set $J^\alpha(\mathfrak{X}_S^k)$ if and only if it can be expressed as

$$X = X^i \left(x, y^{(1)}, \dots, y^{(k)} \right) * \frac{\partial}{\partial y^{(\alpha)i}}, \quad \forall \alpha \in \{0, 1, \dots, k\}.$$

As we have seen, a vector field X on $T^k M \setminus \{0\}$ is a dynamical symmetry if and only if it is a newtonoid and satisfies equations (3.31). Our aim now is to rewrite equations (3.31) such that its terms have a covariant character. Note that neither one of the two terms of equation (3.31), $S^{k+1}(X^i)$ and $X(G^i)$, has such a covariant character. To accomplish this goal, we first characterize newtonoid vector fields in terms of their expressions in the adapted basis (3.10). The components of a newtonoid vector field in the adapted basis (3.10) will behave as the components of a vector field on the base manifold M .

Lemma 3.6. *Consider a semispray S , a nonlinear connection (h_α, v_k) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h_\alpha, v_k))$. A vector field X on $T^k M \setminus \{0\}$ is a newtonoid if and only if*

$$(3.37) \quad X = X^i \frac{\delta}{\delta x^i} + \frac{1}{1!} \nabla^{(1)} X^i \frac{\delta}{\delta y^{(1)i}} + \frac{1}{2!} \nabla^{(2)} X^i \frac{\delta}{\delta y^{(2)i}} + \dots + \frac{1}{k!} \nabla^{(k)} X^i \frac{\delta}{\delta y^{(k)i}},$$

where for $\alpha \in \{1, \dots, k\}$ we denote

$$(3.38) \quad \frac{1}{\alpha!} \nabla^{(\alpha)} X^i = \frac{1}{\alpha!} S^\alpha(X^i) + \sum_{\beta=1}^{\alpha} \frac{1}{(\alpha-\beta)!} M_{(\beta)j}^i S^{\alpha-\beta}(X^j).$$

For a vector field $X \in J^{k-1} \mathfrak{X}(T^k M \setminus \{0\})$ we have that $X \in J^{k-1}(\mathfrak{X}_S^k)$ if and only if

$$v_k(X) = J(\nabla X).$$

Proof. Consider a vector field $X \in \mathfrak{X}(T^k M \setminus \{0\})$. In view of local expression (3.32) and using notation (3.38), we obtain that X is a newtonoid of order k if and only if for $\alpha \in \{1, \dots, k\}$ we have

$$h_\alpha X = \frac{1}{\alpha!} \nabla^{(\alpha)} X^i \frac{\delta}{\delta y^{(\alpha)i}},$$

which is equivalent to formula (3.37).

For a vector field $X \in J^{k-1}(\mathfrak{X}_S^k)$ let $Z \in \mathfrak{X}_S^k$ be such that $X = J^{k-1}Z$. Using formula (3.18) it follows that

$$\begin{aligned} J(\nabla X) &= (J \circ \nabla \circ J^{k-1})Z = (J \circ h_{k-1} \circ \mathcal{L}_S \circ h_{k-1} \circ J^{k-1})Z \\ &= (J \circ h_{k-1} \circ \mathcal{L}_S \circ (J^{k-1} - v_k \circ J^{k-1}))Z. \end{aligned}$$

Using $J \circ h_{k-1} = v_k \circ J$, the fact that for $Z \in \mathfrak{X}_S^k$ we have $(J \circ \mathcal{L}_S \circ J^{k-1})Z = (1-k)J^{k-1}Z$, and the second formula in (3.28) we obtain $J(\nabla X) = v_k(X)$.

Suppose $X = J^{k-1}Z$ for some $Z \in \mathfrak{X}(T^k M \setminus \{0\})$. Starting with $v_k(X) = J(\nabla X)$ then the arguments used to prove the first implication show that $v_k J^k \mathcal{L}_S Z = 0$. Then

$$J^{k-1} \pi_S^k Z = J^{k-1}(\text{Id} + J \mathcal{L}_S)Z = J^{k-1}Z + J^k \mathcal{L}_S Z = X,$$

and $X \in J^{k-1} \text{Im } \pi_S^k = J^{k-1} \mathfrak{X}_S^k$. □

For $\alpha = 1$, formula (3.38) reduces to formula (3.20), which represents the action of the dynamical covariant derivative ∇ on the horizontal components of a vector field. Using formula (3.37) it follows that $\nabla^{(\alpha)} X^i$, which are given by formula (3.38), represent the components of a newtonoid vector field of order k with respect to the basis (3.10). Therefore $\nabla^{(\alpha)} X^i$, behave as the components of a vector field on the base manifold, for arbitrary components X^i that behave as the components of a vector field on the base manifold. This observation will allow us to extend the action of $\nabla^{(\alpha)}$ to arbitrary vector fields on $T^k M \setminus \{0\}$ as follows. Consider a vector field X on $T^k M \setminus \{0\}$ expressed in the basis (3.10) as follows

$$(3.39) \quad X = \sum_{\beta=0}^k X^{(\beta)i} \frac{\delta}{\delta y^{(\beta)i}}.$$

Hence, for each $\beta \in \{0, 1, \dots, k\}$, $X^{(\beta)i}$ behave as the components of a vector field on the base manifold. Accordingly, for each $\alpha \in \{1, \dots, k\}$, the components $\nabla^{(\alpha)} X^{(\beta)i}$ will have the same property. Therefore, we can define the α -th dynamical covariant derivative of X by

$$(3.40) \quad \nabla^{(\alpha)} X = \sum_{\beta=0}^k \nabla^{(\alpha)} X^{(\beta)i} \frac{\delta}{\delta y^{(\beta)i}}, \quad \alpha \in \{1, \dots, k\}.$$

Denote $\nabla^{(0)} = \text{Id}$. For each $\alpha \in \{1, \dots, k\}$, we set $\nabla^{(\alpha)} f = S^\alpha(f)$, for $f \in C^\infty(T^k M \setminus \{0\})$. Then, $\nabla^{(\alpha)}$ satisfies a Leibniz-type rule of order α :

$$\nabla^{(\alpha)}(fX) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \nabla^{(\beta)} f \cdot \nabla^{(\alpha-\beta)} X.$$

Using the Leibniz-type rule of order α , and the requirement that $\nabla^{(\alpha)}$ commutes with tensor contraction, we can extend the action of $\nabla^{(\alpha)}$ to arbitrary tensor fields. For example, consider ω a 1-form on $T^k M \setminus \{0\}$. Then, for an arbitrary vector field X on $T^k M \setminus \{0\}$, we have

$$(3.41) \quad \left(\nabla^{(\alpha)} \omega \right) (X) = S^\alpha(\omega(X)) - \sum_{\beta=1}^{\alpha} \binom{\alpha}{\beta} \left(\nabla^{(\alpha-\beta)} \omega \right) \left(\nabla^{(\beta)} X \right).$$

We emphasize that, for the moment, $\nabla^{(\alpha)}$ may not be the α -th iteration of the dynamical covariant derivative ∇ , unless a specific nonlinear connection will be chosen. In Theorem 3.10 we characterize when $\nabla^{(\alpha)} = \nabla^\alpha$ for all $\alpha \geq 0$.

Proposition 3.7. *Consider the dynamical covariant derivative ∇ associated to a pair $(S, (h_\alpha, v_k))$. A vector field $X \in \mathfrak{X}(T^k M \setminus \{0\})$ is a dynamical symmetry if and only if it is a newtonoid and*

$$(3.42) \quad \frac{1}{k!} \nabla \left(J^k \nabla^{(k)} X \right) + \Phi(X) = 0.$$

Proof. A vector field $X \in \mathfrak{X}(T^k M \setminus \{0\})$ is a dynamical symmetry if and only if $h_\alpha[S, X] = 0$, for all $\alpha \in \{0, \dots, k-1\}$, and $v_k[S, X] = 0$. The first set of conditions $h_\alpha[S, X] = 0$ for all $\alpha \in \{0, \dots, k-1\}$ (or the first k conditions) are equivalent with $[S, X] \in \text{Ker } J$, which is equivalent with $X \in \mathfrak{X}_S^k$.

For any $X \in \mathfrak{X}(T^k M \setminus \{0\})$ we have

$$\nabla(v_k X) + \Phi(X) = v_k[S, X].$$

For any for $X \in \mathfrak{X}_S^k$ (with $h_0 X = X^i \delta / \delta x^i$), formulae (3.37) and (3.40) imply that

$$v_k X = \frac{1}{k!} \nabla^{(k)} X^i \frac{\partial}{\partial y^{(k)i}} = \frac{1}{k!} J^k \nabla^{(k)} X.$$

Using the above observations the result follows. \square

From formula (3.19) we see that the dynamical covariant derivative ∇ has a different action on distributions H_0, \dots, H_{k-1} than on distribution V_k . The next proposition characterizes when these actions coincide. For example, one sufficient condition is that $\nabla J = 0$. When $k = 1$ we know that the canonical nonlinear connection for a semispray is uniquely determined by $\nabla J = 0$. Proposition 3.8 shows that when $k > 1$ there is more freedom, and $\nabla J = 0$ determines only part of a nonlinear connection. To uniquely determine a non-linear connection from a semispray one needs stronger conditions. Such conditions will be given in Theorem 3.10.

Proposition 3.8. *Consider a semispray of order k , S , a nonlinear connection (h_α, v_k) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h_\alpha, v_k))$. Then, the following conditions are equivalent:*

- i) ∇ restricts to a map $\nabla : J^{k-1}(\mathfrak{X}_S^k) \rightarrow J^{k-1}(\mathfrak{X}_S^k)$ that satisfies the Leibniz rule with respect to the $*$ product;
- ii) $\nabla J^\alpha = 0$, for some $\alpha \in \{1, \dots, k\}$;
- iii) $(k+1)v_k \circ J^{k-1} = J^{k-1} + \mathcal{L}_S J \circ J^{k-1}$;
- iv) $N_{(1)j}^i = \partial G^i / \partial y^{(k)j}$;
- v) $\nabla = \nabla^{(1)}$, where $\nabla^{(1)}$ is defined in formula (3.40).

Proof. We first prove that $\nabla J^k = 0$ if and only if $\nabla J^\alpha = 0$, for some $\alpha \in \{1, \dots, k\}$. In view of first formula (3.24) it follows that if $\nabla J^\alpha = 0$, for some $\alpha \in \{1, \dots, k\}$, then $\nabla J^k = 0$. For the converse we will show that $\nabla J^k = 0$ implies $\nabla J = 0$, which will also imply $\nabla J^\alpha = 0$, for all $\alpha \in \{1, \dots, k\}$. From first formula (3.24), we have that $\nabla J^k = \nabla J \circ J^{k-1}$ and therefore the $(1, 1)$ -type tensor field ∇J vanishes on vector fields in $\text{Im } J^{k-1} = V_{k-1}$. From local formulae (3.25) it follows that ∇J vanishes on vector fields in $\bigoplus_{\alpha=0}^{k-2} H_\alpha$, and hence $\nabla J = 0$.

Using the above considerations, formulae (3.24) and (3.25), and the identity $\mathcal{L}_S J^k = \mathcal{L}_S J \circ J^{k-1} - (k-1)J^{k-1}$ we have that conditions *ii*), *iii*), and *iv*) are equivalent.

We prove implication *ii*) \implies *i*). Consider $X \in J^{k-1}(\mathfrak{X}_S^k)$. According to Lemma 3.6, we have that $v_k X = J \nabla X$. If we apply ∇ to both sides of this equality and use $\nabla v_k = 0$ and $\nabla J = 0$ we obtain $v_k(\nabla X) = J \nabla(\nabla X)$, which by Lemma 3.6 implies that $\nabla X \in J^{k-1}(\mathfrak{X}_S^k)$. Hence, ∇ preserves the set $J^{k-1}(\mathfrak{X}_S^k)$.

For $f \in C^\infty(T^k M \setminus \{0\})$ and $X \in J^{k-1}(\mathfrak{X}_S^k)$, formula (3.36) reduces to $f * X = fX + S(f)JX$. Using the fact that ∇ satisfies the Leibniz rule for the usual \cdot product it follows that

$$(3.43) \quad \nabla(f * X) - \nabla(f) * X - f * \nabla X = \nabla(f)(\nabla J)X.$$

Since $\nabla J = 0$ it follows that ∇ satisfies the Leibniz rule for the $*$ product.

We prove now the implication *i*) \implies *ii*). By formula (3.25), which gives the local expression for ∇J^α , we only need to show that $\nabla J = 0$ on vector fields in $\text{Im } J^{k-1}$. A set of generators for vector fields in $\text{Im } J^{k-1}$ is given by $J^{k-1}(\mathfrak{X}_S^k) \cup (\mathfrak{X}^{v_k}(T^k M \setminus \{0\}))$. Since ∇J vanishes on $(\mathfrak{X}^{v_k}(T^k M \setminus \{0\}))$, it remains to show that ∇J vanishes on $J^{k-1}(\mathfrak{X}_S^k)$. Since ∇ satisfies the Leibniz rule for the $*$ product, using formula (3.43) it follows that $S(f)(\nabla J)X = 0$, for an arbitrary function $f \in C^\infty(T^k M \setminus \{0\})$ and an arbitrary vector field $X \in J^{k-1}(\mathfrak{X}_S^k)$. Therefore, $\nabla J = 0$ on the set $J^{k-1}(\mathfrak{X}_S^k)$.

Using formulae (3.19), (3.38), and (3.40) we have that $\nabla^{(1)} = \nabla$ if and only if $\nabla J = 0$. Hence conditions *ii*) and *v*) are equivalent. \square

Up to this point, the structure of Section 3 follows the structure of Section 2, and each result for a system of SODE has a counterpart for a system of HODE. Due to the complications imposed by the geometry of a system of HODE we will have results that are specific only to the higher order case.

Although Proposition 3.8 fixes only a part of the nonlinear connection, we can obtain now an explicit covariant form for the equations of variation (3.31), given by the following Jacobi-type equations (3.44).

Proposition 3.9. *Consider ∇ the dynamical covariant derivative associated to a pair $(S, (h_\alpha, v_k))$, and suppose that $\nabla J = 0$. Then, a vector field X on $\mathfrak{X}(T^k M \setminus \{0\})$ is a dynamical symmetry if and only if X is a newtonoid and*

$$(3.44) \quad \frac{1}{k!} \nabla \left(\nabla^{(k)} X^i \right) + \sum_{\alpha=0}^{k-1} \frac{1}{\alpha!} R_{(\alpha)j}^i \nabla^{(\alpha)} X^j = 0,$$

where X^i are the horizontal components of X such that $h_0 X = X^i \delta / \delta x^i$.

Proof. If X is a newtonoid, then formulae (3.14) and (3.37) imply that

$$\Phi(X) = \sum_{\alpha=0}^{k-1} \frac{1}{\alpha!} R_{(\alpha)j}^i \nabla^{(\alpha)} (X^j) \frac{\partial}{\partial y^{(k)i}},$$

and the result follows by Proposition 3.7. \square

Note that while using formula (3.42) in Proposition 3.7, to characterize a newtonoid X on $T^k M \setminus \{0\}$, we essentially need the assumption $\nabla J = 0$. This implies that the two actions of ∇ in formulae (3.19) coincide. Therefore, the action of ∇ on the components of the vertical vector field $v_k X$ is given by the second formula (3.20).

Next theorem will completely determine the nonlinear connection and this will allow us to provide simpler expressions for the k curvature components, $R_{(\alpha)j}^i$, of the Jacobi endomorphism Φ .

Theorem 3.10. *Consider a semispray of order k , S , a nonlinear connection (h_α, v_k) , and the dynamical covariant derivative ∇ associated to the pair $(S, (h_\alpha, v_k))$. Then, the following conditions are equivalent:*

- i) $\mathcal{L}_S J + \text{Id} - (k+1)v_k = i_J \Phi$;
- ii) $\nabla^{(\alpha)} = \nabla^\alpha$, for all $\alpha \in \{1, 2, \dots, k\}$;
- iii) $M_{(1)i}^j = \partial G^j / \partial y^{(k)i}$ and $\alpha M_{(\alpha)j}^i = S \left(M_{(\alpha-1)j}^i \right) + M_{(\alpha-1)j}^p M_{(1)p}^i$, for all $\alpha \in \{2, \dots, k\}$;
- iv) $N_{(1)i}^j = \partial G^j / \partial y^{(k)i}$ and $\alpha N_{(\alpha)j}^i = S \left(N_{(\alpha-1)j}^i \right) - N_{(\alpha-1)p}^i N_{(1)p}^j$, for all $\alpha \in \{2, \dots, k\}$.

Proof. Using formulae (3.14) and (3.26) we obtain that equality i) holds true if and only if $\nabla J = 0$ and for each $\alpha \in \{1, \dots, k-1\}$ we have

$$(3.45) \quad R_{(\alpha)j}^i = (k+1) \left(\frac{\delta G^i}{\delta y^{(\alpha)j}} - N_{(k+1-\alpha)j}^i \right).$$

In view of formulae (3.17), for α in $\{1, \dots, k-1\}$, above equations (3.45) are equivalent to

$$(3.46) \quad (k+1-\alpha) N_{(k+1-\alpha)j}^i = S(N_{(k-\alpha)j}^i) - N_{(1)j}^s N_{(k-\alpha)s}^i - \sum_{\beta=2}^{k-\alpha} I_{(\beta)j}^s N_{(k+1-\alpha-\beta)}^i.$$

When $\alpha \in \{1, \dots, k-1\}$, equality (3.46) is equivalent with $I_{(k-\alpha+1)j}^i = 0$. Hence condition i) holds if and only if $\nabla J = 0$ and $I_{(\alpha)j}^i = 0$ for all $\alpha \in \{2, \dots, k\}$. Induction and equation (3.15) show that the latter conditions are equivalent with the second formulae in iv).

Let us first note that condition ii) is equivalent with $\nabla J = 0$ and $\nabla^{(\alpha)} = \nabla \circ \nabla^{(\alpha-1)}$ for all $\alpha \in \{2, \dots, k\}$. For $\alpha \in \{1, \dots, k\}$, and $\beta \in \{0, \dots, k\}$, equations (3.38) and (3.40) imply that

$$\nabla^{(\alpha)} \frac{\delta}{\delta y^{(\beta)j}} = \alpha! M_{(\alpha)j}^i \frac{\delta}{\delta y^{(\beta)i}},$$

and when $\alpha \in \{2, \dots, k\}$ we have

$$\left(\nabla \circ \nabla^{(\alpha-1)} \right) \frac{\delta}{\delta y^{(\beta)j}} = (\alpha-1)! \left(S \left(M_{(\alpha-1)j}^i \right) + M_{(\alpha-1)j}^p M_{(1)p}^i \right) \frac{\delta}{\delta y^{(\beta)i}}.$$

It follows that ii) and iii) are equivalent.

Using formulae (3.12) and (3.41), and induction for $\alpha \in \{1, \dots, k\}$, it follows that $\nabla^{(\alpha)}$ has the following action on the dual basis (3.11)

$$\nabla^{(\alpha)} \delta y^{(\beta)i} = -\alpha! N_{(\alpha)j}^i \delta y^{(\beta)j}, \quad \beta \in \{0, 1, \dots, k\}.$$

From this formula, using the action of the dynamical covariant derivative ∇ on 1-forms, we obtain, for each $\beta \in \{0, 1, \dots, k\}$,

$$\left(\nabla \circ \nabla^{(\alpha-1)} \right) \delta y^{(\beta)i} = (\alpha-1)! \left(S \left(N_{(\alpha-1)j}^i \right) - N_{(\alpha-1)p}^i N_{(1)j}^p \right) \delta y^{(\beta)j}.$$

In view of the above two formulae it follows that condition ii) for $\alpha \in \{2, \dots, k\}$ and the second formulae in condition iv) are equivalent. \square

The $(1, 1)$ -type tensor $\mathcal{L}_S J - \text{Id} + (k+1)v_k$ in the first item of Theorem 3.10 can be used to characterize some other nonlinear connections. For example if we require it to vanish, we obtain a nonlinear connection studied in [1, 6, 18, 20].

For a semispray S of order k , we say that the nonlinear connection characterized by any of the four conditions in Theorem 3.10 is called the *canonical nonlinear connection* of S . For the remaining of this section we will consider this canonical nonlinear connection induced by a semispray S . The dual coefficients $M_{(\alpha)j}^i$ of this connection were considered first by Miron and Atanasiu in [36]. Next theorem characterizes the symmetries of a semispray S using the dynamical covariant derivative ∇ associated to a semispray S and its canonical nonlinear connection.

Theorem 3.11. *Let (h_α, v_k) be the canonical nonlinear connection induced by a semispray S , and let ∇ be the corresponding dynamical covariant derivative.*

- i) *A vector field X on $T^k M \setminus \{0\}$ is a dynamical symmetry if and only if X is a newtonoid and*

$$(3.47) \quad \frac{1}{k!} \nabla^{k+1} J^k X + \Phi(X) = 0.$$

- ii) *A vector field $X \in \mathfrak{X}(M)$ is a Lie symmetry if and only if*

$$(3.48) \quad \frac{1}{k!} \nabla^{k+1} X^{k,0} + \Phi(X^{k,k}) = 0.$$

- iii) *Locally, a newtonoid vector field X on $T^k M \setminus \{0\}$ is a dynamical symmetry of S if and only if its horizontal components satisfy the Jacobi equation*

$$(3.49) \quad \frac{1}{k!} \nabla^{k+1} X^i + \sum_{\alpha=0}^{k-1} \frac{1}{\alpha!} R_{(\alpha)j}^i \nabla^\alpha X^j = 0.$$

iv) The components of the Jacobi endomorphism are given by

$$(3.50) \quad \begin{aligned} R_{(\alpha)j}^i &= (k+1) \left(\frac{\delta G^i}{\delta y^{(\alpha)j}} - N_{(k+1-\alpha)j}^i \right), \quad \alpha \in \{1, \dots, k-1\} \\ R_{(0)j}^i &= (k+1) \frac{\delta G^i}{\delta x^j} - S \left(N_{(k)j}^i \right) + N_{(k)l}^i N_{(1)j}^l. \end{aligned}$$

Proof. Part i) follows by combining Proposition 3.7 and Theorem 3.10. Part ii) follows from Part i) using the fact that $J^k X^{k,k} = X^{k,0}$. Part iii) represents the local expression of Part i). In part iv), the first formula in (3.50) was proven in the proof of Theorem 3.10. The second formula in (3.50) follows by setting $\alpha = 0$ and $I_{(2)j}^i = 0, \dots, I_{(k-1)j}^i = 0$ in formula (3.17). \square

3.3. First order variation for systems of HODE. Let $c : I \rightarrow M$ be a geodesic of a semispray S of order k . A *geodesic variation* of c is a smooth map $V : I \times (-\varepsilon, \varepsilon) \rightarrow M$, $V = V(t, s)$ such that

- i) $V(t, 0) = c(t)$, for all t in I ,
- ii) $V(t, s)$ is a geodesic for all s in $(-\varepsilon, \varepsilon)$.

Definition 3.12. A vector field $\xi : I \rightarrow TM$, along a geodesic $c = \pi \circ \xi$ of a semispray S , is called a *Jacobi field* of S if it is the variation vector field $\xi(t) = \partial_s V(t, s)|_{s=0}$ of a geodesic variation V .

Next we provide a sufficient condition for a vector field $\xi : I \rightarrow TM$ to be a Jacobi field. This sufficient condition generalizes the traditional Jacobi equation to a semispray of order k .

For a vector field $\xi : I \rightarrow TM$, $\xi(t) = (x^i(t), \xi^i(t))$, along a regular curve $c = \pi \circ \xi$, and $\alpha \in \{0, \dots, k\}$, we define the lifted vector fields $\xi^{k,\alpha} : I \rightarrow T(T^k M \setminus \{0\})$, along the k -th jet lift $j^k c$ of c , as follows

$$\xi^{k,\alpha} = \left(x^i(t), \frac{1}{1!} \frac{dx^i}{dt}(t), \dots, \frac{1}{k!} \frac{d^k x^i}{dt^k}(t), 0, \dots, 0, \xi^i(t), \frac{1}{1!} \frac{d\xi^i}{dt}(t), \dots, \frac{1}{\alpha!} \frac{d^\alpha \xi^i}{dt^\alpha}(t) \right).$$

Since ξ is a vector field along c , for each $t \in I$, we can find a vector field $X \in \mathfrak{X}(M)$ such that $X \circ c = \xi$ near t . Then $\xi^{k,\alpha} = X^{k,\alpha} \circ j^k c$, for all $\alpha \in \{0, \dots, k\}$. We assume now that c is a geodesic, which means that $j^k c$ is an integral curve of a semispray S . For $\alpha \geq 0$ we define

$$(3.51) \quad \nabla^\alpha \xi^{k,0} = (\nabla^\alpha X^{k,0}) \circ j^k c.$$

If locally $\xi = (x^i, \xi^i)$ and $X = X^i \partial / \partial x^i$, then $\nabla \xi^{k,0} = \nabla \xi^i \partial / \partial y^{(k)i}$, where

$$(3.52) \quad \nabla \xi^i = (\nabla X^i) \circ j^k c = \frac{d\xi^i}{dt} + N_{(1)j}^i \left(x, \frac{1}{1!} \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) \xi^j.$$

By induction on α we find that

$$\nabla^\alpha \xi^i = (\nabla^\alpha X^i) \circ j^k c, \quad \alpha \geq 0,$$

where $\nabla^\alpha X^i$ are iterations of formula (3.20), that correspond to formulae (3.38) for the nonlinear connection determined by Theorem 3.10. Hence, using $\nabla^\alpha X^{k,0} = \nabla^\alpha X^i \partial / \partial y^{(k)i}$,

$$\nabla^\alpha \xi^{k,0} = \nabla^\alpha \xi^i \frac{\partial}{\partial y^{(k)i}}, \quad \alpha \geq 0.$$

It follows that definition (3.51) does not depend on the choice of X , and components $\nabla^\alpha \xi^i$ transform as a tensor on M .

Proposition 3.13. Let ξ be a vector field $\xi : I \rightarrow TM$ along a geodesic of a semispray S of order k . If ξ is a Jacobi field of S then it satisfies the following Jacobi equation

$$(3.53) \quad \frac{1}{k!} \nabla^{k+1} \xi^{k,0} + \Phi(\xi^{k,k}) = 0.$$

If locally $\xi(t) = (x^i(t), \xi^i(t))$, equation (3.53) is equivalent to

$$(3.54) \quad \frac{1}{k!} \nabla^{k+1} \xi^i + \sum_{\alpha=0}^{k-1} \frac{1}{\alpha!} R_{(\alpha)j}^i \left(x, \frac{1}{1!} \frac{dx}{dt}, \dots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) \nabla^\alpha \xi^j = 0.$$

Proof. By induction on α we have that

$$\delta y^{(\alpha)i} \xi^{k,k} = \frac{1}{\alpha!} \nabla^\alpha \xi^i, \quad \alpha \in \{0, \dots, k\}.$$

Hence formulae (3.53) and (3.54) are equivalent. To prove that formula (3.53) holds, let $c: I \rightarrow M$ be the geodesic $c = \pi \circ \xi$ of S , let $t \in I$, and let X be a vector field $X \in \mathfrak{X}(M)$ such that $X \circ c = \xi$ on some interval $I_0 \ni t$. Since ξ is a Jacobi field there is a geodesic variation $V(t, s)$ such that $V(t, 0) = c(t)$, $\partial V^i / \partial s(t, 0) = \xi^i(t)$, and

$$\frac{\partial^{k+1} V^i}{\partial t^{k+1}} + (k+1)! G^i \left(V, \frac{1}{1!} \frac{\partial V}{\partial t}, \dots, \frac{1}{k!} \frac{\partial^k V}{\partial t^k} \right) = 0, \forall (t, s) \in I_0 \times (-\varepsilon, \varepsilon).$$

Differentiating the last equation with respect to s and setting $s = 0$ gives

$$\frac{d^{k+1} \xi^i}{dt^{k+1}} + (k+1)! \left(\frac{\partial G^i}{\partial x^j} \xi^j + \frac{\partial G^i}{\partial y^{(1)j}} \frac{1}{1!} \frac{d\xi^j}{dt} + \dots + \frac{\partial G^i}{\partial y^{(k)j}} \frac{1}{k!} \frac{d^k \xi^j}{dt^k} \right) = 0.$$

Since c is a geodesic, it follows that $S^\alpha(f) \circ j^k(c) = d^\alpha(f \circ j^k c) / dt^\alpha$ for all $\alpha \geq 0$ and $f \in C^\infty(T^k M \setminus \{0\})$. Thus

$$(S^{k+1}(X^i) + (k+1)! X^{k,k}(G^i)) \circ j^k c = 0.$$

By Lemma 3.3 and inclusion (3.35) it follows that $J[S, X^{k,k}] = 0$. Then

$$v_k[S, X^{k,k}] = \frac{1}{k!} S^{k+1}(X^i) + X^{k,k}(G^i),$$

and $[S, X^{k,k}] \circ j^k c = 0$. Repeating the argument in Proposition 3.7 shows that

$$\frac{1}{k!} \nabla^{k+1} (J^k X^{k,k}) \circ j^k c + \Phi(X^{k,k}) \circ j^k c = 0.$$

Equation (3.53) follows. \square

A proof for the converse of Proposition 3.13 will require to extend all the techniques developed in [11]. Another aspect that will have to be addressed in the future is the role of the curvature components $R_{(\alpha)j}^i$ for the geodesic behavior.

4. APPLICATIONS

In this section we motivate the applicability of the geometric theory developed in this paper using examples from various fields such as: the equivalence problem, the inverse problem of the calculus of variations, and biharmonicity.

First, we consider a second order Lagrangian, L_2 , [9] derived from a Riemannian structure. The corresponding Euler-Lagrange equations form a system of fourth order differential equations, whose solutions are biharmonic curves [13]. For this system of differential equations, the components of the Jacobi endomorphism are functions of the curvature of the Riemannian metric and its dynamical covariant derivatives. This will motivate the use of *curvature components* for the components of the Jacobi endomorphism.

We also show that the curvature components of the Jacobi endomorphism are useful to express geometric invariants that were associated previously to third or fourth order ordinary differential equations. See [20, 21, 22, 38].

4.1. Prolongation of a Riemannian structure. In this section we start with a Riemannian space (M, g) and construct a 2-nd order Lagrangian $L_2 \in C^\infty(T^2M)$ [9]. The Euler-Lagrange equations for L_2 determine a system of fourth order ordinary differential equations and hence a semispray of order 3. Solutions of this system are biharmonic curves for the Riemannian space [13]. The 3 components $R_{(\alpha)j}^i$, $\alpha \in \{0, 1, 2\}$, of the Jacobi endomorphism can be expressed in terms of the curvature components R_{jkl}^i and their first and second order dynamical covariant derivatives.

Consider $g = g_{ij}(x)dx^i \otimes dx^j$ a semi-Riemannian metric on M and denote by $\gamma_{jk}^i(x)$ its Christoffel symbols. The geodesic spray of the Riemannian metric g has the following components of the Jacobi endomorphism $R_j^i(x, y^{(1)}) = R_{kjl}^i(x)y^{(1)k}y^{(1)l}$. We denote

$$(4.1) \quad z^{(2)i} = y^{(2)i} + \frac{1}{2}\gamma_{jk}^i(x)y^{(1)j}y^{(1)k}.$$

It follows that $z^{(2)i}$ behave as the components of a vector field on M . These components were interpreted as the covariant form of acceleration in [9, (6.5)] and as half of the components of the tension field in [13]. Therefore, the function $L_2 : T^2M \rightarrow \mathbb{R}$, given by

$$(4.2) \quad L_2(x, y^{(1)}, y^{(2)}) = \frac{1}{2}g_{ij}z^{(2)i}z^{(2)j},$$

is a second order Lagrangian, [9, 13]. The variational problem for L_2 leads to the following Euler-Lagrange equations

$$(4.3) \quad \frac{\partial L_2}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial L_2}{\partial y^{(1)i}} \right) + \frac{1}{2} \frac{d^2}{dt^2} \left(\frac{\partial L_2}{\partial y^{(2)i}} \right) = 0,$$

which form a system of fourth order ordinary differential equations. Consider $S \in \mathfrak{X}(T^3M)$

$$(4.4) \quad S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + 3y^{(3)i} \frac{\partial}{\partial y^{(2)i}} - 4G^i \frac{\partial}{\partial y^{(3)i}},$$

the corresponding semispray of order 3. This semispray is uniquely determined by the following equation

$$(4.5) \quad \frac{\partial L_2}{\partial x^i} - S \left(\frac{\partial L_2}{\partial y^{(1)i}} \right) + \frac{1}{2} S^2 \left(\frac{\partial L_2}{\partial y^{(2)i}} \right) = 0.$$

Indeed, the functions G^i , of the semispray S , uniquely determined by equation (4.5), are given by

$$(4.6) \quad 6g_{ji}G^j = \frac{\partial L_2}{\partial x^i} - d_T \left(\frac{\partial L_2}{\partial y^{(1)i}} \right) + \frac{1}{2} d_T^2 \left(\frac{\partial L_2}{\partial y^{(2)i}} \right),$$

where d_T is the Tulczyjew operator on T^3M . Although, above expression for the coefficients G^i of the semispray S is not so easy to handle with, we will show that one can obtain the corresponding nonlinear connection, which is determined by any of the four equivalent conditions in Theorem 3.10. In formula (4.5) we apply $\partial/\partial y^{(3)k}$, use the fact that the Lagrangian L_2 and its partial derivatives are functions on T^2M we obtain

$$\frac{\partial G^i}{\partial y^{(3)j}} = N_{(1)j}^i(x, y^{(1)}) = M_{(1)j}^i(x, y^{(1)}) = \gamma_{kj}^i(x)y^{(1)k}.$$

For the other two coefficients $N_{(2)j}^i(x, y^{(1)}, y^{(2)})$, $N_{(3)j}^i(x, y^{(1)}, y^{(2)}, y^{(3)})$, of the nonlinear connection we use formulae iv) from Theorem 3.10.

Consider ∇ the dynamical covariant derivative associated to the semispray S and the corresponding nonlinear connection. It follows that

$$z^{(2)i} = \frac{1}{2} \nabla y^{(1)i},$$

hence the name of *covariant form of acceleration* for $z^{(2)i}$. Equation (4.5) is then equivalent to

$$(4.7) \quad \frac{\delta L_2}{\delta x^i} - \nabla \left(\frac{\delta L_2}{\delta y^{(1)i}} \right) + \frac{1}{2} \nabla^2 \left(\frac{\delta L_2}{\delta y^{(2)i}} \right) = 0.$$

Using the following formulae

$$\frac{\delta z^{(2)i}}{\delta y^{(1)j}} = 0, \quad 2g^{ik} \frac{\delta L}{\delta x^k} = R_k^i z^{(2)k}, \quad 2 \left(\frac{\delta z^{(2)i}}{\delta x^j} + \gamma_{jk}^i z^{(2)k} \right) = R_j^i,$$

we can rewrite equation (4.7) as follows

$$(4.8) \quad \nabla^2 z^{(2)i} + R_j^i z^{(2)j} = 0.$$

Note that this equation is equivalent to

$$\nabla^3 y^{(1)i} + R_j^i \nabla y^{(1)j} = 0.$$

Therefore the Euler-Lagrange equations (4.3) can be written as follows

$$\nabla^3 \left(\frac{dx^i}{dt} \right) + R_j^i \nabla \left(\frac{dx^j}{dt} \right) = 0,$$

which is the system of differential equations for biharmonic curves [13, (1)]. Since a geodesic of the Riemannian metric g is a solution of the system $\nabla(dx^i/dt) = 0$, it follows that any geodesic is a biharmonic curve. Conversely, if the sectional curvature is non-positive then any biharmonic closed curve is a geodesic, see [26]. However, if we change the parameter of an arbitrary geodesic by a third order polynomial, the new curve is no longer a geodesic but a biharmonic curve, see [3].

Using any of the equivalent form (4.8), (4.6), or (4.7), of equations (4.5), we can express the components $R_{(\alpha)j}^i$, $\alpha \in \{0, 1, 2\}$ of the Jacobi endomorphism in terms of the curvature tensor R_{jkl}^i of the Riemannian metric. Two of them are

$$\begin{aligned} 3R_{(2)j}^i &= 2R_j^i = 2R_{sjk}^i y^{(1)s} y^{(1)k}, \\ 3R_{(1)j}^i &= \nabla R_j^i + R_{jsk}^i \nabla y^{(1)s} y^{(1)k}, \end{aligned}$$

The other component $R_{(0)j}^i$, has a more complicated formula, but it is a function of the curvature tensor R_{jkl}^i , its first, and second order dynamical derivative.

4.2. The Wuenschmann invariant. A geometric invariant that relates third order ordinary differential equations with certain classes of conformal Lorentz metrics on three dimensional manifolds, was proposed by K. Wuenschmann in his Ph.D thesis of 1905. The role of the Wuenschmann invariant for the geometry of a third order differential equation under contact transformations was discussed in [38]. See also [21] for a generalization of the problem to fourth order ordinary differential equations, where the equivalence problem leads to three invariants. A geometric method for obtaining the Wuenschmann invariant for a system of third order ordinary differential equations was proposed in [20]. In this section we show that the Wuenschmann invariant can be obtained from the components of the Jacobi endomorphism.

On a 1-dimensional manifold M , consider a third-order differential equation

$$\frac{d^3 x}{dt^3} + 3!G \left(x, \frac{1}{1!} \frac{dx}{dt}, \frac{1}{2!} \frac{d^2 x}{dt^2} \right) = 0,$$

and the corresponding semispray S of order 2. There exists an associated conformal Lorentzian structure on the 3-dimensional solution space of this equation if and only if an invariant vanishes,

[20, 23]. This invariant is called the *Wuenschmann invariant* and it is given by [38, eq. (4)]

$$W_3 = -\frac{1}{2}S^2\left(\frac{\partial G}{\partial y^{(2)}}\right) - 3\frac{\partial G}{\partial y^{(2)}}S\left(\frac{\partial G}{\partial y^{(2)}}\right) + 3S\left(\frac{\partial G}{\partial y^{(1)}}\right) - 2\left(\frac{\partial G}{\partial y^{(2)}}\right)^3 + 6\frac{\partial G}{\partial y^{(1)}}\frac{\partial G}{\partial y^{(2)}} - 6\frac{\partial G}{\partial x}.$$

Consider the two curvature components given by formula (3.50) of the Jacobi endomorphism:

$$\begin{aligned} R_{(0)} &= 3\frac{\partial G}{\partial x} - 3\frac{\partial G}{\partial y^{(1)}}\frac{\partial G}{\partial y^{(2)}} - \frac{1}{2}S^2\left(\frac{\partial G}{\partial y^{(2)}}\right) + \left(\frac{\partial G}{\partial y^{(2)}}\right)^3, \\ R_{(1)} &= 3\frac{\partial G}{\partial y^{(1)}} - \frac{3}{2}\left(\frac{\partial G}{\partial y^{(2)}}\right)^2 - \frac{3}{2}S\left(\frac{\partial G}{\partial y^{(2)}}\right). \end{aligned}$$

It follows that the Wuenschmann invariant and the components of the Jacobi endomorphism are related as follows

$$(4.9) \quad W_3 = \nabla R_{(1)} - 2R_{(0)}.$$

It is important to note that same formula (4.9) was obtained in [20] for two other different nonlinear connections.

4.3. The inverse problem of Lagrangian mechanics for scalar fourth-order ordinary differential equations. In [22], Fels has shown that a scalar fourth-order ordinary differential equation admits a variational multiplier if and only if two invariants vanish. These two geometric invariants were associated with a fourth-order equation using Cartan's equivalence method. One of these invariants appears also in the list of invariants proposed by Dridi and Neut in [21] for studying the equivalence problem for fourth order differential equations under fiber preserving diffeomorphisms. In this section we show that this invariant can be expressed in terms of the components of the Jacobi endomorphism using a very similar formula as in the previous section.

On a 1-dimensional manifold M , consider a fourth-order differential equation

$$\frac{d^4x}{dt^4} + 4!G\left(x, \frac{1}{1!}\frac{dx}{dt}, \frac{1}{2!}\frac{d^2x}{dt^2}, \frac{1}{3!}\frac{d^3x}{dt^3}\right) = 0,$$

and the corresponding semispray S of order 3. Consider the following Wuenschmann-type invariant

$$W_4 = -\frac{2}{3}S^2\left(\frac{\partial G}{\partial y^{(3)}}\right) - 4\frac{\partial G}{\partial y^{(3)}}S\left(\frac{\partial G}{\partial y^{(3)}}\right) + 4S\left(\frac{\partial G}{\partial y^{(2)}}\right) - \frac{8}{3}\left(\frac{\partial G}{\partial y^{(3)}}\right)^3 + 8\frac{\partial G}{\partial y^{(2)}}\frac{\partial G}{\partial y^{(3)}} - 8\frac{\partial G}{\partial y^{(1)}}.$$

The invariant W_4 is related to the invariant I_1 considered by Fels in [22] as follows $W_4 = -3I_1$. Also $(8/3)W_4$ is the numerator of I_9 in [21].

Consider two curvature components $R_{(1)}$ and $R_{(2)}$ given by formula (3.50) of the Jacobi endomorphism:

$$\begin{aligned} R_{(1)} &= 4\frac{\partial G}{\partial y^{(1)}} - 4\frac{\partial G}{\partial y^{(2)}}\frac{\partial G}{\partial y^{(3)}} - \frac{2}{3}S^2\left(\frac{\partial G}{\partial y^{(3)}}\right) + \frac{4}{3}\left(\frac{\partial G}{\partial y^{(3)}}\right)^3, \\ R_{(2)} &= 4\frac{\partial G}{\partial y^{(2)}} - 2\left(\frac{\partial G}{\partial y^{(3)}}\right)^2 - 2S\left(\frac{\partial G}{\partial y^{(3)}}\right). \end{aligned}$$

It follows that the invariant W_4 and the components of the Jacobi endomorphism are related as follows

$$(4.10) \quad W_4 = \nabla R_{(2)} - 2R_{(1)}.$$

We note the similarities for expressing the two invariants W_3 and W_4 in formulae (4.9) and (4.10).

As an example consider the classical spinning particle [31]. The motion of a particle rotating about a translating center is governed by the following system of fourth order differential equations

$$(4.11) \quad \frac{d^4 x^i}{dt^4} + \omega^2 \frac{d^2 x^i}{dt^2} = 0, \quad i \in \{1, 2, 3\},$$

where ω is a real, non-zero constant. For this system of fourth order we have

$$G^i = \frac{1}{12} \omega^2 y^{(2)i}, \quad N_{(1)j}^i = N_{(2)j}^i = N_{(3)j}^i = 0.$$

The only non-vanishing component of the Jacobi endomorphism is

$$R_{(2)j}^i = \frac{1}{3} \omega^2 \delta_j^i.$$

Therefore, the Wuenschmann invariant is $W_4 = \nabla R_{(2)} - 2R_{(1)} = 0$. Note that the system (4.11) is separable and hence we can view each of the three equations of the system separately. For each of them, the Wuenschmann invariant vanishes and therefore each equation is a Lagrangian equations. A second order Lagrangian for the system (4.11) is the following

$$L(x, y^{(1)}, y^{(2)}) = \delta_{ij} y^{(2)i} y^{(2)j} - \frac{\omega^2}{2} \delta_{ij} y^{(1)i} y^{(1)j}.$$

For the system (4.11), the Jacobi equations are

$$\nabla^4 \xi^i + \omega^2 \nabla^2 \xi^i = 0,$$

which can be integrated. Therefore we can obtain all geodesic variations of the system (4.11).

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IOAN BUCATARU, OANA CONSTANTINESCU; FACULTY OF MATHEMATICS, AL.I.CUZA UNIVERSITY, B-DUL CAROL
11, IASI, 700506, ROMANIA

URL: <http://www.math.uaic.ro/~bucataru/>

URL: <http://www.math.uaic.ro/~oanacon/>

MATIAS F. DAHL; INSTITUTE OF MATHEMATICS, AALTO UNIVERSITY, FI-00076 AALTO, HELSINKI, FINLAND

URL: <http://www.math.tkk.fi/~fdahl/>